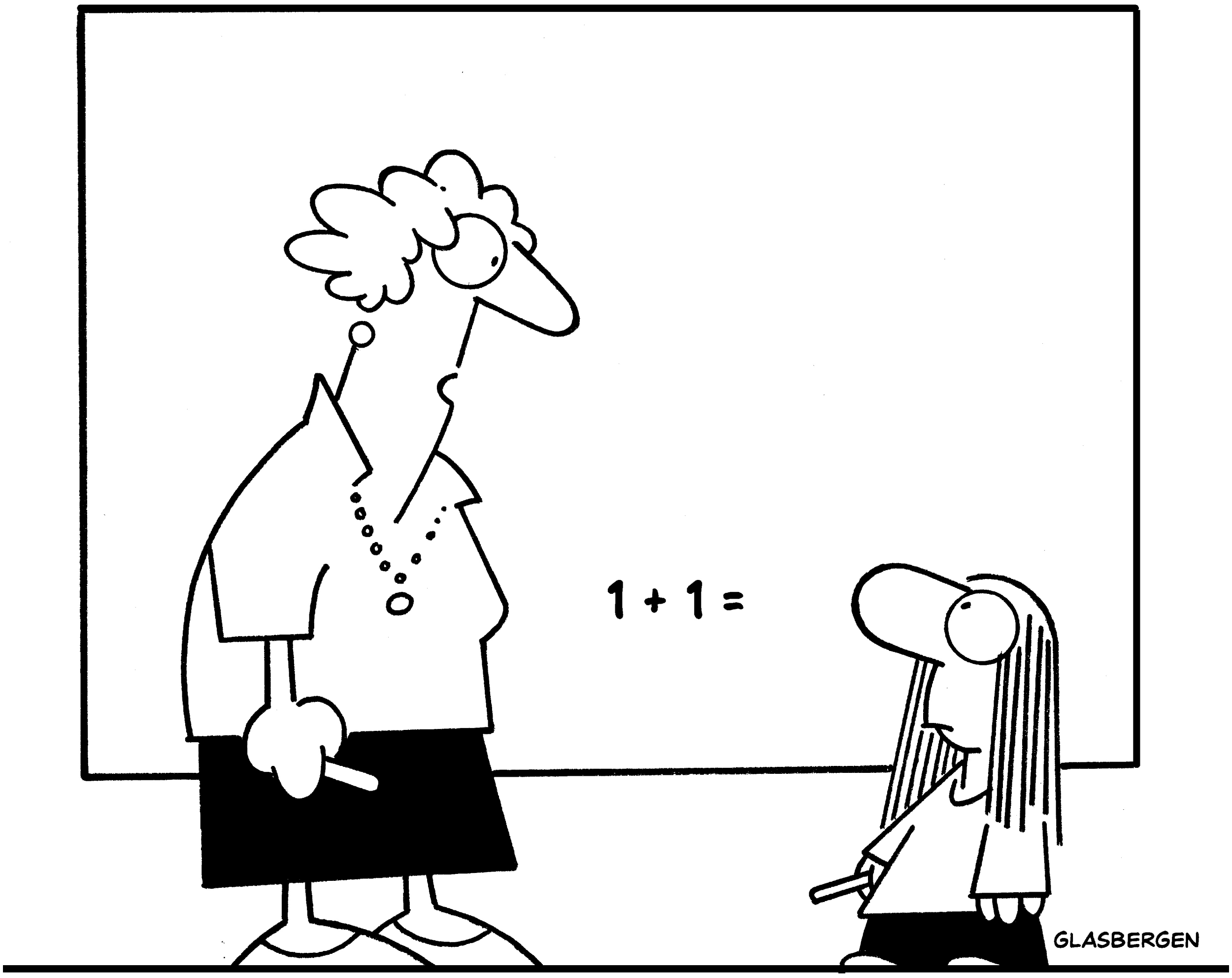
Mathematics Preparatory Course

for Engineers, Scientists and Economists



“Yes, this will be useful to you later in life.”

Preface to the third edition

Dear Beginning Students,

This script was created for the mathematics portion of the preparatory courses for studies at the Ulm University of Applied Sciences. Our goal is to help you recognize possible gaps and to help you fill these. In addition, it should serve as an aid in digging up forgotten knowledge and refreshing it. Perhaps you will then discover exactly like we have done that math is simply terrific and actually a lot of fun.

Dependent on your course of studies, individual chapters in the preparatory course can be disregarded. Relevant exercises can be found in Part 2 and the corresponding solutions in Part 3. Further exercises which are not addressed during the preparatory course are available for self-study subsequent to the preparatory course. Your tutors will decide how intensely the individual sections of the chapter will be dealt with.

Should you find you have additional practice requirements or if you simply like to study online, you will find a **Mathematics Training Course** as well as a **Differential Calculus Training Course** in our learning management system under the path *Fakultäten → Mathematik, Natur- und Wirtschaftswissenschaften → Allgemein*. You can read up on all the topics here and complete many interactive practice exercises. This can also serve as preparation for the **Mathematics Entry Exam**. A further possibility for practice and verification is the **Minimum Requirements Catalog**. You can find the catalog under the link *http://www.mathematik-schule-hochschule.de* with the materials.

This script is based on many individual scripts which were created for the preparatory courses at the Ulm University of Applied Sciences. We would like to sincerely thank all the authors for their suggestions and the exercises we have utilized. At the end of each chapter there are optional exercises for experienced students which are somewhat trickier and time-consuming to solve or may not have been addressed directly in the script. Principally, not all exercises include a solution process because there is often not a single ideal path which leads to the solution. In addition, searching for errors in one's own calculation method is beneficial for furthering one's own understanding and for exposing errors in reasoning. If you find yourself getting bogged down, please contact one of the tutors.

We hope you enjoy the preparatory course and that you do well!

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Last modified in August 2019.
Should you discover an error in the script, I would be very happy if you would report it to thorsten.titzmann@thu.de.

Contents

Part I.

Theory

1. Fundamentals

This chapter is intended to help you fill any gaps in fundamental arithmetic operations and mathematical principles. For your studies, these fundamentals are part of the basic tool set which you absolutely must be able to competently apply in the technical and scientific fields of study. You should self-study those portions of this chapter which are not addressed in the precourse in order to acquire their content, if in fact you have not already (securely) mastered them.

1.1. Arithmetic

As a university graduate with a technical background, you must possess a very good understanding of numbers. This understanding is especially important since you must be capable of intuitively determining the plausibility of your own results as well as those of others. These abilities are especially essential if your calculations for your work are performed nearly exclusively with the support of computers.In order to develop a good understanding of numbers as well as judgment capability, mental arithmetic (also with written support) is the first step. Doing math without a calculator also helps you understand fundamental mathematical concepts, eliminate any potentially incorrect perceptions, and achieve competent handling of mathematical term transformations (especially arithmetic with fractions, roots, powers, solving of equations). The latter, in turn, are the foundation of mathematics and all applications derived thereof (for example, economics, physics, computer science, mechanical engineering, quantitative methods, etc.). If you should determine that your arithmetic skills have gotten rusty, you can rejuvenate them through simple practice. The progress you make through practicing will make it increasingly easier for you to calculate independently.

To start off, it is recommended to work on the arithmetic training (6.1) for the fundamental arithmetic operations.

Long Division

The method of long division is also required for polynomial division, which will be described at a later point in the script. But first of all here is an

Example:

- $2346 : 3 = 782$
  The corresponding arithmetic looks like this:

$$\begin{array}{rrrrr|l}
 & 2 & 3 & 4 & 6 & 3 \\ \cline{6-6}
- & \underline{2} & \underline{1} & & & 7\ 8\ 2 \\
 & & 2 & 4 & & \\
 & - & \underline{2} & \underline{4} & & \\
 & & & 0 & 6 & \\
 & & - & & \underline{6} & \\
 & & & 0 & 0 &
\end{array}$$

  The principle behind this method proceeds according to the following procedure:

- In order to divide numbers, they are written next to each other separated by the division symbol.

- ◦ The first digit of the number on the left is divided by the number on the right. If that is not possible, the second number on the left is considered as well, in this case this would be 23.
- ◦ Now multiply the result with the number on the right. Write the product underneath the digits used from the left-hand number and find the difference.
  Note: Be certain and line up the numbers correctly.
- ◦ Then bring down the next digit from the number on the left and calculate again.
- ◦ Follow this procedure until all the digits of the number on the left-hand side have been used.
  If the last difference is zero, the calculation is completed.

- If the last step of the division results in a non-zero difference and the number on the left has no more digits, than a zero is added to the left-hand number and a decimal point is inserted in the result. Then the next digit of the result is calculated, or the difference is noted as a remainder..

$$134 : 4 = 33 \text{ R2 } = 33\ 5$$

The corresponding calculation is here:

$$
\begin{array}{ccccc|l}
 & 1 & 3 & 4 & & 4 \\ \cline{6-6}
- & 1 & 2 & & & 3\,3{,}5 \\ \cline{2-3}
 & & 1 & 4 & & \\
 & - & 1 & 2 & & \\ \cline{3-4}
 & & & 2 & 0 & \\
 & & - & 2 & 0 & \\ \cline{4-5}
 & & & 0 & 0 &
\end{array}
$$

- In some cases, the division keeps producing the same digits in the result after a few steps. This is called periodic.

$$2 : 7 = 0.\overline{285714}$$

$$
\begin{array}{cccccccc|l}
 & 2 & & & & & & & 7 \\ \cline{9-9}
- & 0 & & & & & & & 0{,}2\,8\,5\,7\,1\,4 \\ \cline{2-2}
 & 2 & 0 & & & & & & \\
- & 1 & 4 & & & & & & \\ \cline{2-3}
 & & 6 & 0 & & & & & \\
 & - & 5 & 6 & & & & & \\ \cline{3-4}
 & & & 4 & 0 & & & & \\
 & & - & 3 & 5 & & & & \\ \cline{4-5}
 & & & & 5 & 0 & & & \\
 & & & - & 4 & 9 & & & \\ \cline{5-6}
 & & & & & 1 & 0 & & \\
 & & & & - & & 7 & & \\ \cline{6-7}
 & & & & & & 3 & 0 & \\
 & & & & & - & 2 & 8 & \\ \cline{7-8}
 & & & & & & & 2 &
\end{array}
$$

Mental Arithmetic

When performing mental arithmetic, your brain must typically perform two functions:

Calculating and **Remembering**

In case you are inexperienced, relieve the burden on your brain by not requiring it to perform the remembering function until you have progressed by practicing: At the beginning of your arithmetic training, take written notes (intermediate results, etc.). You will find that after a short time you can do without the memory aid and calculate fluently..

Examples:

1. You would like to calculate 23^2. There are various approaches which you could apply. These all are essentially individual multiplications with the subsequent addition of the intermediate results. In the initial phase of your training you can write down the intermediate results and add them at the end. Later on, you will realize that you no longer require any paper for such calculations.

   - Approach 1: $20 \cdot 23 = 460$ and $3 \cdot 20 = 60$ and $3 \cdot 3 = 9$ leads to
   $$23^2 = 460 + 60 + 9 = 529$$
   - Approach 2: Squares of numbers can easily be calculated using the first binomial equation: $(a+b)^2 = a^2 + 2ab + b^2$ leads to
   $$23^2 = (20+3)^2 = 20^2 + 2 \cdot 20 \cdot 3 + 3^2 = 400 + 120 + 9 = 529$$

2. You would like to calculate the decimal representation of $\frac{7}{3}$. This can be done in your head the same way as when you are writing down division, only that you need to remember the intermediate results. Hint: To begin with, write down the final write successively while you are solving for it:
   $$\frac{7}{3} = 2 + remainder$$
   You write down the 2 and remember the remainder 1
   $$\frac{10}{3} = 3 + remainder$$
   You write down the 3: 2.3 The remainder is 1 again and thus it becomes clear that all that is left to do is to draw the periodicity line over the 3:
   $$\frac{7}{3} = 2.\overline{3}$$

3. Later on, try to go without writing anything down: You would like to calculate the decimal representation of $\frac{11}{7}$:
   $$\frac{11}{7} = 1 + remainder$$
   You remember the 1 and the remainder 4.
   $\frac{40}{7} = 5+$ remainder, you remember 1.5 and the remainder 5.
   $\frac{50}{7} = 7+$ remainder, you remember 1.57 and the remainder 1.
   $\frac{10}{7} = 1+$ remainder, you remember 1.571 and the remainder 3.
   $\frac{30}{7} = 4+$ remainder, you remember 1.5714 and the remainder 2.
   $\frac{20}{7} = 2+$ remainder, you remember 1.57142 and the remainder 6.
   $\frac{60}{7} = 8+$ remainder, you remember 1.571428 and the remainder 4.

Since $\frac{40}{7}$ gives you 5 plus remainder, you now know that you have the periodically repeating 571428.

$$\frac{11}{7} = 1,\overline{571428}$$

Hint: If a division by 7 is not possible without a remainder, the result is always a periodically repeating 14 28 57. This is simple to remember since 14 is two times 7, $28 = 2 \cdot 14$ and $57 = 2 \cdot 28 + 1$

4. When you calculate fractions, reduce first in order to keep the multiplications and divisions as simple as possible. In this case as well you will need to remember the intermediate results of the reduction.

$$\frac{3}{7} \cdot \frac{56}{6}$$

You can reduce 3 with 6 and 56 with 7. What remains is $\frac{1}{1} \cdot \frac{8}{2} = 4$.
In your head, always only reduce a few numbers and remember the new intermediate result. Then reduce again and proceed as before:

$$\frac{3}{7} \cdot \frac{56}{6} = \frac{56}{7 \cdot 2} = \frac{8}{2} = 4$$

5. You would like to calculate 6 3% of 120,000: First of all, reduce $1\% = \frac{1}{100}$ with 120,000: What remains is $6\ 3 \cdot 1200$ You immediately know: $6 \cdot 1200 = 7200$ and $0\ 3 \cdot 1200$ is then half of one tenth of the last intermediate result:$0\ 3 \cdot 1200 = \frac{720}{2} = 360$. The solution is thus

$$6\ 3\% \cdot 120{,}000 = 7200 + 360 = 7560$$

1.2. Mathematical Operations and Transformations

For $a, b, c \in \mathbb{R}$ the following applies:

1. Commutative laws

$$a + b = b + a; \qquad a \cdot b = b \cdot a$$

2. Associative laws

$$(a + b) + c = a + (b + c); \qquad (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

3. Distributive law **Remember: multiplication/division before addition/subtraction**

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

4. Dealing with negative signs

$$(-a)b = -ab = a(-b); \quad (-a)(-b) = ab; \quad -(-a) = a; \quad a - (b + c) = a - b - c$$

Multiplication of Sums and Factoring

Two sums are multiplied with one another by multiplying every component of the first sum with every component of the second sum, taking all signs into account while doing so.

- $(a + b) \cdot (c + d) = ac + ad + bc + bd$

- $(a + b) \cdot (c - d) = ac - ad + bc - bd$
- $(a - b) \cdot (c - d) = ac - ad - bc + bd$
- $(a + b + c) \cdot (2x - y - z) = 2ax - ay - az + 2bx - by - bz + 2cx - cy - cz$

If all the components of a sum have a common factor, the factor can be taken out and placed in front of parentheses, thus transforming the sum into a product..

- $ax + bx + cx = x \cdot (a + b + c)$
- $a \cdot (x + 1) + b \cdot (x + 1) = (a + b) \cdot (x + 1)$
- $2ab - a - 4ac = -a \cdot (-2b + 1 + 4c)$
- $2ax + 2ay + 3bx + 3by = 2a \cdot (x + y) + 3b \cdot (x + y) = (2a + 3b) \cdot (x + y)$

Binomial Equation

1. First binomial equation: $(a + b)^2 = a^2 + 2ab + b^2$
2. Second binomial equation: $(a - b)^2 = a^2 - 2ab + b^2$
3. Third binomial equation: $(a + b) \cdot (a - b) = a^2 - b^2$

1.3. Calculating Fractions

In fractions, the following mathematical operations can be performed without changing the value of the fraction:

1. Reduction: Numerator and denominator are divided by the same number ($\neq 0$).
2. Expansion: Numerator and denominator are multiplied by the same number ($\neq 0$).

Important: Both the entire numerator and the entire denominator must be divided/multiplied.

Examples:

- $\frac{a}{a} = 1,\ \frac{0}{a} = 0,\ a \neq 0$
- $\frac{abxy}{ab} = xy \qquad a, b \neq 0$
- $\frac{ab + ac}{a} = \frac{a \cdot (b + c)}{a} = b + c \qquad a \neq 0$
- $\frac{ax - bx + ay - by}{a - b} = \frac{x \cdot (a - b) + y \cdot (a - b)}{a - b} = \frac{(x + y) \cdot (a - b)}{a - b} = x + y$

Addition and Subtraction

Fractions with the same denominator are said to be similar or to share a **common denominator**. If fractions have different denominators, they are called dissimilar. Fractions with a common denominator are added/subtracted by adding/subtracting the numerators and keeping the denominator the same.

$$\frac{a}{d} + \frac{b}{d} + \frac{c}{d} = \frac{a + b + c}{d} \qquad d \neq 0$$

Dissimilar fractions must be expanded so that they have a common denominator before they can be added/subtracted.

If in doubt, a common denominator can always be found by using the product of the two different denominators!

Examples:

1. $\frac{5a}{2x} + \frac{3a}{8x} = \frac{20a}{8x} + \frac{3a}{8x} = \frac{20a + 3a}{8x} = \frac{23a}{8x}$
2. $\frac{7a}{12b} - \frac{3a}{5c} = \frac{7a \cdot 5c}{12b \cdot 5c} - \frac{3a \cdot 12b}{5c \cdot 12b} = \frac{35ac - 36ab}{60bc}$

Multiplication and Division

Two fractions can be multiplied by multiplying the respective numerators and denominators with each other.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad b, d \neq 0$$

Two fractions can be divided by multiplying the first fraction by the reciprocal of the second fraction.

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \qquad b, c, d \neq 0$$

1.4. Exponent and Root Arithmetic, Logarithms

Arithmetic with exponents makes it possible to transform multiplication and division operations into addition and subtraction. This is a substantial simplification and is applied constantly in calculations. Many operations are significantly easier with exponents (rather than fractions or roots).

- $\underbrace{a \cdot a \cdot \ldots \cdot a}_{n-\text{mal}} = a^n \qquad a = \text{base},\ n = \text{exponent}$
- $a^0 = 1$
- $a^1 = a$
- $(-a)^n = \begin{cases} a^n & \text{if n is even} \\ -a^n & \text{if n is odd} \end{cases}$

Addition and Subtraction

Only exponents with the same base and same exponent can be added or subtracted.

Examples:

- $3x^3 - 5x^3 + 4x^3 = 2x^3$
- $6a^2b^3 + 4a^2 - 2b^3 + 7a^2 = 6a^2b^3 + 11a^2 - 2b^3$

Multiplication and Division

Arithmetic rule	Example
$a^m \cdot a^n = a^{m+n}$	$4x^3y^7 \cdot 5x^4y^2 = 20x^{3+4}y^{7+2} = 20x^7y^9$
$a^n \cdot b^n = (a \cdot b)^n$	$2x^{5y} \cdot 5z^{5y} = (2x \cdot 5z)^{5y} = (10xz)^{5y}$
$\frac{a^m}{a^n} = a^{m-n}$	$\frac{12x^3y^4}{3x^2y^2} = 4x^{3-2}y^{4-2} = 4xy^2$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{2x}{y}\right)^{4z} = \frac{(2x)^{4z}}{y^{4z}}$
$a^{-n} = \frac{1}{a^n}$	$3x^{-2}y^3 = \frac{3y^3}{x^2}$
$(a^n)^m = a^{m \cdot n}$	$(a^5)^{2x} = a^{10x}$
Special case:	$b^0 = 1$ and $b^{-r} = \frac{1}{b^r}$

Roots

In addition, the **root sign** or **radical** is defined by:

$$x = \sqrt[n]{a} \Leftrightarrow x^n = a \quad (a \geqslant 0)$$

$$a^{\frac{1}{n}} = \sqrt[n]{a} \qquad a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$$

This also clarifies exponent arithmetic for rational exponents.

Logarithms

If there is a variable in the exponent, the equation is called an exponential equation, for example: $4^{2x} = 256$
Equations of the following type are solved by applying the logarithm.

$$a^x = b \Leftrightarrow x = \log_a b$$
$$2^x = 512 \Leftrightarrow x = \log_2 512 = 9$$
$$5^x = \frac{1}{125} \Leftrightarrow x = \log_5 \frac{1}{125} = -3$$

The logarithm of base 10 is also called the **decadic** or **common logarithm** and is designated by the symbol lg:

$$\log_{10} a = \lg a$$

The logarithm of base e is also called the **natural logarithm** and is designated by the symbol ln , where $e \approx 2\ 72$ is Euler's number. It is commonly used in science and in engineering.

$$\log_e a = \ln a$$

Every possible logarithm $\log_a b$ can be represented by the natural logarithm:

$$\log_a b = \frac{\ln b}{\ln a}$$

Examples:

- $\log_2 8 = 3;\qquad \log_2 1024 = 10;\qquad \log_{10} 100 = 2;\qquad \log_{10} \frac{1}{100} = -2$

Arithmetic rules

- $\log_a 1 = 0$
- $\log_a a = 1$
- $a^{\log_a b} = b$
- $\log_a (u \cdot v) = \log_a u + \log_a v$
- $\log_a \frac{u}{v} = \log_a u - \log_a v$
- $\log_a u^v = v \cdot \log_a u$

1.5. Numbers and Sets

Georg Cantor (March 3, 1845 - January 6, 1918), gave the following definition of a set in 1895 :

Definition 1.1:

"Under a 'set' we understand every collection M of determined, properly distinguished objects m of our intuition or our thought (which are designated as the 'elements' of M) into a whole."

This definition requires that all objects (also called **elements**) of a set are uniquely definable, that they are all distinguishable, and especially that no element is included more than once. The elements do not need to be numbers.

Examples:

- The set of natural numbers from 1 to 5: $M = \{1,2,3,4,5\}$
- The set with the single element 3: $M = \{3\}$
- The empty set contains no elements: $M = \{\} = \emptyset$
- A vertical line (or a colon), which is pronounced as "such that" separates rules from the naming of the elements:

$$M = \{x|\, x \text{ is a square} < 30\} = \{1,4,9,16,25\}$$

- If rules are not realizable, this can lead to the empty set:

$$M = \{x|\, x \in \mathbb{R} \text{ and } x^2 + 1 = 0\} = \{\}$$

Notation with sets

1. $x \in M$ means: x is an element M
2. $x \notin M$ means: x is not an element of M
3. $M_1 = M_2$ means: The set M_1 is identical with setM_2: $x \in M_1 \Leftrightarrow x \in M_2$ and $x \notin M_1 \Leftrightarrow x \notin M_2$
4. $M_1 \subset M_2$ or $M_1 \subseteq M_2$ means: SetM_1 is a subset of set M_2: $x \in M_1 \Rightarrow x \in M_2$ and $x \notin M_2 \Rightarrow x \notin M_1$
5. $A = M_1 \cup M_2$ is the union of set M_1 with set M_2. A contains all elements which are contained inM_1 OR ($\vee$) M_2.

6. $A = M_1 \cap M_2$ is the intersection of sets M_1 and M_2. A contains all elements which are contained inM_1 AND ($\wedge$) M_2.
7. $A = M_1 \backslash M_2 = \{x | x \in M_1 \wedge x \notin M_2\}$ is the difference set: M_1 reduced by M_2. A contains all element which are contained in M_1 AND NOT in M_2. Careful: $M_1 \backslash M_2 \neq M_2 \backslash M_1$
8. $A = M_1 \times M_2 = \{(x, y) | x \in M_1,\ y \in M_2\}$ is the Cartesian product of the sets M_1 and M_2.

Examples:

- $2 \in \{1,2,3\}$, $4 \notin \{1,2,3\}$
- Let $M_1 = \{1,2,3,4\}$, $M_2 = \{4,5,6,7\}$, $M_3 = \{4,5\}$, then:
    - $M_3 \subset M_2$
    - $M_1 \cap M_2 = \{4\}$
    - $M_2 \backslash M_1 = \{5,6,7\}$
- $x \in M \backslash \{0\}$ is pronounced as: "x is an element of M without 0"

The sets$\mathbb{N}, \mathbb{Z}$

The set of the **natural numbers** is:

$$\mathbb{N} = \{1,2,3,4....\}$$

The set of all even natural numbers for example can be described by:

$$M = \{2x |\ x \in \mathbb{N}\} = \{2,4,6\ldots\}$$

$\mathbb{N}_0$ describes the union of the (set of) natural numbers and zero:

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

The expansion of number ranges was usually motivated by the unsolvability of certain equations. Beginning with $\mathbb{N}$, numerous expansions are possible. The set of **whole numbers** is:

$$\mathbb{Z} = \mathbb{N} \cup \{0, -1, -2, -3....\}$$

Examples:

- $2x = 16$ has exactly one solution in $\mathbb{N}$
- $-2x = 6$ has no solution in $\mathbb{N}$, but exactly one solution in $\mathbb{Z}$
- $0 \cdot x = 0$, has infinite solutions in $\mathbb{Z}$
- $3x = 1$ cannot be solved in $\mathbb{Z}$

The set $\mathbb{Q}$

If an equation is solvable in A, then it is also solvable in B if $A \subset B$ applies. In order to solve an equation such as $3x = 1$, an expansion to include the set of **rational numbers** is required:

$$\mathbb{Q} = \left\{ \frac{m}{n} | \, m \in \mathbb{Z}, \, n \in \mathbb{N} \right\}$$

Satz 1.1. *Every rational number is a periodic or finite decimal number. And every periodic or finite decimal number is rational.*

Examples:

- $r \cdot x = s$, has for $r \neq 0$ exactly one solution $x \in \mathbb{Q}$ for fixed $r, s \in \mathbb{Q}$
- $x \cdot x = 2$, has no solution in $\mathbb{Q}$
- $\frac{2}{7} = 0.\overline{285714}$ (Note: The number of remainders for the division $\frac{m}{n}$ is $\leqslant n - 1$.)
- $x = 2\ 12\overline{345}$
  $100x = 212.\overline{345}$ and $100{,}000x = 212{,}345.\overline{345}$
  $99{,}900x = 212{,}133 \Rightarrow x = \dfrac{212133}{99900}$

The set $\mathbb{R}$

The number 0.1 10 100 1000 10000 100000..... cannot be contained in$\mathbb{Q}$ since it is neither periodic nor finite. Besides this number constructed here, we know many other numbers which cannot be contained in $\mathbb{Q}$. One of these numbers is $x = \sqrt{2}$. The corresponding equation $x^2 = 2$ has no solution in $\mathbb{Q}$.

One possibility to define the set of **real numbers** is $\mathbb{R} = \{x | x \text{ is a decimal number}\}$

Every point on the number line is contained in $\mathbb{R}$.

Interval notation and inequalities

For $a \leqslant b$ the following notations apply:

$$[a, b] := \{x | a \leqslant x \leqslant b\}$$
$$(a, b) := \{x | a < x < b\}$$

$$(a, b] := \{x | a < x \leqslant b\}$$
$$[a, \infty) := \{x | a \leqslant x\}$$
$$(-\infty, b) := \{x | x < b\}$$

The round parentheses can also be replaced by square brackets open to the outside:

$$(a, b) =]a, b[\, , \ [a, b) = [a, b[\text{ etc.}$$

Segments of the number line are described by intervals or inequalities. The set of all positive real numbers corresponds to the right half of the number line. This segment of the set $\mathbb{R}$ can be described by an inequality or an interval:

$$[0, \infty) = \{x \in \mathbb{R} | 0 \leqslant x\}$$

Comments:

- An interval may also contain only a single point: $[a, a] = a$
- Non-empty intersections of intervals are again intervals.
- The solutions to inequalities are intervals or unions of intervals.

If two values are connected by an inequality symbol, **inequality**. The same transformations as applied to an equation can be applied in order find the solution. If multiplied or divided by a factor $c < 0$, $c \in \mathbb{R}$, the inequality symbol is inverted. That means that the following equivalency transformations apply.

1. Addition or subtraction of the same value on both sides of the equation.
2. Multiplication or division of both sides of the equation by a factor $c \neq 0$.
   a) $c > 0$: the inequality symbol remains unchanged.
   b) $c < 0$: the inequality symbol is inverted.

Example:

$$\begin{array}{rcll} & -7x & \leqslant & 4x + 18 \\ \Leftrightarrow & -11x & \leqslant & 18 \\ \Leftrightarrow & x & \geqslant & -\frac{18}{11} \\ \Leftrightarrow & L & = & \left\{x \in \mathbb{R} \mid x \geqslant -\frac{18}{11}\right\} \\ \Leftrightarrow & L & = & \left[-\frac{18}{11}, \infty\right) \end{array}$$

1.6. Quantities and Units

Measurement is one of the most important tasks in physics and engineering. Besides the required measuring equipment, standardized units are needed which are grouped together in a single system. With very few exceptions, the “International System of Units (SI Units)“ is now used. The following table displays the seven base units:

Quantity	**Unit**	**Symbol**
length	meter	m
time	second	s
mass	kilogram	kg
electric current	ampere	A
temperature	kelvin	K
amount of substance	mol	mol
luminous intensity	candela	cd

The remaining SI units are derived as powers, products, or quotients of the base units coherently, that is without use of numerical factors. All other units are incoherent and thus not SI units.

Examples:

1. Watt [W] is a coherent unit for power, that is, derived without numerical factors:

$$1\mathrm{W} = 1\frac{\mathrm{kg} \cdot \mathrm{m}^2}{\mathrm{s}^3}$$

2. Kilowatt [kW] is an incoherent unit for power, that is, derived with the use of a numerical factor:
$$1\text{kW} = 10^3 \cdot \frac{\text{kg} \cdot \text{m}^2}{\text{s}^3}$$

The following table shows which numerical factor is required relative to a base unit for each respective prefix.

Prefix	**Symbol**	**Factor**	**Prefix**	**Symbol**	**Factor**
tera	T	10^{12}	deci	d	10^{-1}
giga	G	10^{9}	centi	c	10^{-2}
mega	M	10^{6}	milli	m	10^{-3}
kilo	k	10^{3}	micro	μ	10^{-6}
hecto	h	10^{2}	nano	n	10^{-9}
deca	da	10			

For example, a milliampere mA is one thousandth of an Ampere, i.e. $1\text{mA} = 1 \cdot 10^{-3}\text{A}$.

1.7. Cross-Multiplication

This solution method is used when certain values are in a proportional relationship to one another..

Example:

A car uses 6.8 ℓ of fuel for a distance of 100 km. How much fuel does the car require for a distance of 35 km?

First we calculate the fuel consumption for one kilometer:

$$6\ 8\ \ell : 100\text{ km} = 0\ 068\ \ell/\text{km}$$

We can then calculate the consumption for a distance of 35 km:

$$0\ 068\ \ell/\text{km} \cdot 35\text{ km} = 2\ 38\ \ell$$

Schematically this can be displayed as follows:

$$\begin{array}{rr} 100\text{ km} & 6.8\text{ l} \\ :100 \downarrow & :100 \downarrow \\ 1\text{ km} & 0.068\text{ l} \\ \cdot 35 \downarrow & \cdot 35 \downarrow \\ 35\text{ km} & 2.38\text{ l} \end{array}$$

If different parameters are inversely proportional to one another, then the inverse of the arithmetic rule needs to be applied.

Example:

Two painters need 5 days to paint a house. How long do 5 painters need to paint the house if all of them work at the same speed?

Two painters need 5 days to paint the house. That means that one painter needs

$$2 \cdot 5\text{ days} = 10\text{ days}$$

That means 5 painters are faster than 2 since

$$10 \text{ days} : 5 = 2 \text{ days}$$

They need only 2 days to paint the house:

$$\begin{array}{rr} 2 \text{ painter} & 5 \text{ days} \\ :2 \downarrow & \cdot 2 \downarrow \\ 1 \text{ painter} & 10 \text{ days} \\ \cdot 5 \downarrow & :5 \downarrow \\ 5 \text{ painter} & 2 \text{ days} \end{array}$$

1.8. Systems of Linear Equations

A system of linear equations is a system of linear equations which contains multiple unknown variables.
Basic techniques to solve such systems are:

1. Substitution method
2. Elimination method
3. Augmentation method

All three methods basically function on the same principle: They reduce the number of unknowns and the number of equations step by step until the first unknown is determined. All further unknowns are then solved for recursively.

Example:

All three methods solve the system of linear equations: $3x + 2y = 9$ and $4x - y = 1$ uniquely with $x = 1$ and $y = 3$:

1. **Substitution method:** The second equation is solved for y and inserted into the first equation:

$$\begin{aligned} y &= 4x - 1 \\ \Rightarrow 3x + 2(4x - 1) &= 9 \\ \Leftrightarrow 11x &= 11 \Rightarrow x = 1 \end{aligned}$$

   Substituting into the second equation results in: $y = 4 \cdot 1 - 1 \Rightarrow y = 3$

2. **Elimination method:** The first and second equation are solved for y and set equal to one another:

$$\begin{aligned} y &= \frac{1}{2}(9 - 3x) \\ y &= 4x - 1 \\ \Rightarrow \frac{1}{2}(9 - 3x) &= 4x - 1 \\ 9 - 3x &= 8x - 2 \Rightarrow x = 1 \end{aligned}$$

   Inserting into the second equation results in: $y = \frac{1}{2}(9 - 3) \Rightarrow y = 3$

3. **Augmentation method:** Here we add double the second equation to the first equation:

$$\begin{aligned} 3x + 2y &= 9 \\ 4x - y &= 1 \\ \Rightarrow 3x + 2y + 2(4x - y) &= 9 + 2 \\ 3x + 8x &= 11 \Rightarrow x = 1 \end{aligned}$$

Insertion into one of the equations results in: $y = 3$

Solution set of a system of linear equations

Systems of linear equations can have exactly one, none, or an infinite number of solutions. The methods shown above can be applied to systems of linear equations with any number of variables or equations.

Examples:

1. The system of linear equations:

$$\begin{aligned} 3x + 2y &= 9 \\ 4x - y &= 1 \\ 2x - 2y &= 3 \end{aligned}$$

has no solution: From the first two equations we obtain: $x = 1$ and $y = 3$. However, this does not solve the third equation. This system of linear equations therefore has no solution.

2. The following system of linear equations has an infinite number of solutions:

$$\begin{aligned} &x + y + z = 4 \\ &x + y - z = -8 \end{aligned}$$

Subtracting the first from the second equation (we multiply by (-1) and add the result), gives us:

$$-2z = -12 \Leftrightarrow z = 6$$

Adding the first to the second equation results in:

$$2x + 2y = -4 \Leftrightarrow x + y = -2 \Leftrightarrow y = -x - 2$$

Since we have now exhausted all options and both equations are fulfilled, y can only be determined dependent on x. This means that the solution set can be described as follows:

$$\begin{aligned} &x \in \mathbb{R} \text{ arbitrary} \\ &y = -x - 2 \\ &z = 6 \end{aligned}$$

2. Functions

Definition 2.1:
A real-valued function f of a real variable assigns every value of $x \in D \subseteq \mathbb{R}$ exactly one value $f(x) \in \mathbb{R}$.

The **domain** of a function is part of the definition of the function**.** If the domain is not specified, then the largest possible defined set (in $\mathbb{R}$) is meant.

Functions are **maps**, they map their **domain** to the **range of values**. This is designated by:

$$f : D \to W$$

$f(x)$ represents the value of the function at point x. The common expression "function $f(x) = ...$" is strictly speaking imprecise. What is meant is: "The function f with domain D and the mapping rule $f(x) = ...$".

the **graph of a function** is a **set of points**: $G_f = \{(x, f(x))|x \in D\}$

Example

The functions $f : (0{,}17] \to \mathbb{R}$ with $f(x) = \frac{1}{x}$ and $g : \mathbb{R}^- \to \mathbb{R}$ with $g(x) = \frac{1}{x}$ have the same mapping rule, yet quite evidently are very different from one another.

Roots

Definition 2.2:
If a function $f : \mathbb{R} \to \mathbb{R}$ has the value $f(x_0) = 0$ at a point$x_0 \in \mathbb{R}$, then x_0 is a **root**, or **zero**, of function f.

Zeros are those points at which the graph of the function crosses or is tangential to the x-axis.

2.1. Linear Functions

Definition 2.3:
A function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = ax + b,\ a, b \in \mathbb{R}$ is called **linear**.

The graphs of these functions are always straight lines with slope a. The y-intercept is given by $f(0) = b$.

Two-point equation and point-slope equation

If two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ of a line are given, the equation for the line can be determined by applying the **gradient triangle**:

$$a = \frac{f(x) - f(x_1)}{x - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The point-slope equation is also obtained from this basic relationship.

Examples:

1. Determine the equation for the line through the points $P_1 = (2, -3)$ and $P_2 = (-1{,}4)$.

$$\Rightarrow \frac{f(x) - (-3)}{x - 2} = \frac{4 - (-3)}{-1 - 2} \Leftrightarrow f(x) + 3 = -\frac{7}{3}(x - 2) \Leftrightarrow f(x) = -\frac{7}{3}x + \frac{5}{3}$$

2. Determine the equation for the line with slope $a = 2$ that passes through point $P = (1, -3)$

$$\Rightarrow 2 = \frac{f(x) - (-3)}{x - 1} \Leftrightarrow f(x) + 3 = 2x - 2 \Leftrightarrow f(x) = 2x - 5$$

Tip: Familiarize yourself with these relationships for the **gradient triangles of lines**. Make certain that you are able to apply them intuitively.

Zeros of linear functions

The zeros of linear functions are solutions to linear equations. They can be calculated using simple mathematical transformations.

2.2. Quadratic Functions

Functions of type $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = ax^2 + bx + c,\ a, b, c \in \mathbb{R}$ are also called quadratic functions. The corresponding graphs are parabolas. In order to be able to evaluate the graphs of these functions, it is recommended to transform them into the **vertex form** $f(x) = a(x - B)^2 + C$ by completing the square.

Completing the square generally refers to transforming an equation with the form $a^2 + 2ab$ to an equation of form $a^2 + 2ab + b^2$in order to produce the right-hand side of a binomial equation:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Method: First, b is determined and $b^2$is subsequently added. In order not to change the complete equation, b^2 is then immediately subtracted again. The conversion to the vertex form therefore proceeds as in the following:

$$\begin{aligned} f(x) = ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\ &= a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2\right) - a\left(\frac{b}{2a}\right)^2 + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

with the vertex point $S = \left(-\frac{b}{2a},\, c - \frac{b^2}{4a}\right) = (B,\, C)$.

Example:

$$f(x) = x^2 + 4x - 2 = (x + 2)^2 - 6 \Rightarrow S = (-2, -6)$$

Note:

If the vertex point lies at $S = (0{,}0)$ the parabola is similar to the unit parabola $f(x) = ax^2$ but stretched by a factor a in the direction of the $f(x)$-axis. For $a = 1$ this is exactly the unit parabola. For further information see also: 2.9

Tip: Be certain that you are able to spontaneously sketch any given parabola at any given time.

Zeros of quadratic functions

Roots, or zeros, of quadratic functions are solutions to quadratic equations. They can be found by **completing the square** as was the case above.

If the real set of solutions of a quadratic equation is being sought, than all $x \in \mathbb{R}$ which solve the equation: $ax^2 + bx + c = 0$ are being sought. To simplify the process, division by a is recommended. (This is possible since $a \neq 0$ applies.)

$ax^2 + bx + c = 0 \Leftrightarrow x^2 + px + q = 0$ where $p = \frac{b}{a}$ and $q = \frac{c}{a}$

Completing the square in the above equation and then solving for x results in the $pq-$equation:

$$x^2 + px + q = 0$$

$$x^2 + 2\frac{p}{2}x = -q$$

$$x^2 + 2\frac{p}{2}x + \left(\frac{p}{2}\right)^2 = -q + \left(\frac{p}{2}\right)^2$$

$$\left(x + \frac{p}{2}\right)^2 = \frac{p^2}{4} - q$$

$$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - q} \Rightarrow x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

The radicand $\rho := \dfrac{p^2}{4} - q$ is also often called the **discriminant** because this equation makes it possible to immediately make statements about the solution set:

$$\begin{aligned}
\rho > 0 \quad &\Rightarrow \quad \text{2 unique solutions, 2 roots} \\
&\Rightarrow \quad \text{The vertex point lies below the } x\text{-axis} y_S < 0 \\
\rho = 0 \quad &\Rightarrow \quad \text{1 (double) solution, 1 root} \\
&\Rightarrow \quad \text{The vertex point lies on the } x\text{-axis: } y_S = 0 \\
\rho < 0 \quad &\Rightarrow \quad \text{No (real) solution, no real roots} \\
&\Rightarrow \quad \text{The vertex point lies above the } x\text{-axis: } y_S > 0
\end{aligned}$$

Examples:

1. $x^2 - 3x - 4 = 0 \Rightarrow x_{1,2} = -\dfrac{-3}{2} \pm \sqrt{\dfrac{(-3)^2}{4} - (-4)} = \dfrac{3}{2} \pm \sqrt{\dfrac{9}{4} + \dfrac{16}{4}} \Rightarrow x_1 = 4,\ x_2 = -1$
2. $x^2 - 4x + 4 = 0 \Rightarrow x_{1,2} = 2 \pm \sqrt{\dfrac{16}{4} - \dfrac{16}{4}} \Rightarrow x_1 = 2$
3. $x^2 - 2x + 3 = 0 \Rightarrow x_{1,2} = 1 \pm \sqrt{1 - 3} \Rightarrow$ no real solution

2.3. Rational Functions (Polynomials)

Definition 2.4:
A function $p : \mathbb{R} \to \mathbb{R}$ is called a **polynomial** (more specifically: a polynomial function), when an explicit description of the function by

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1}x^{n-1} + a_n x^n = \sum_{k=0}^{n} a_k x^k$$

with real numbers $a_0, a_1, \ldots, a_n$ exist.

- An expression with form $a_k x^k$ is called a **monomial and is also a polynomial.**
- The **degree of the polynomial** is $n \in \mathbb{N}_0$ (condition: $a_n \neq 0$)
- The a_k are called **coefficients**.
- Linear functions are polynomials of degree one: $p(x) = a_0 + a_1 x$
- Quadratic functions are polynomials of degree two: $p(x) = a_0 + a_1 x + a_2 x^2$

Examples:

Write down the degree of the following polynomials.

1. $p(x) = x^3 + x + 5$
2. $p(x) = 7$
3. $p(x) = (x-2)^2(x+3)$

Polynomial division

Polynomial division, also called partial division, is performed analogously to division of numbers with remainders but instead of two numbers, two polynomials are divided by one another and the result is also a polynomial - the “whole“ part and the remainder of the division.

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_n x^n \qquad a_0, ..., a_n \in \mathbb{R};\ n \in \mathbb{N}$$

The highest degree n determines the degree of the polynomial. Each polynomial of degree n possesses a maximum of n real roots. Thus the polynomial can be represented as follows:

$$P(x) = (x - x_1) \cdot (x - x_2) \cdot ... \cdot (x - x_n) \qquad n \in \mathbb{N}$$

The values $x_1, x_2, ..., x_n$ are the roots of the polynomial.

Determining the roots of polynomials of degree zero or one is very simple. For polynomials of degree two, the $pq-$equation represents a very effective possibility for determining the roots. As of degree three, however, it becomes more difficult to calculate the roots exactly.

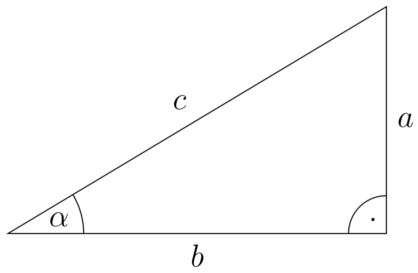
Assume the case of a polynomial $P(x)$ for which one root x_1 is known. The quotient $\frac{P(x)}{(x-x_1)}$ now provides a polynomial of a lower degree. In the case of a polynomial of degree three, a quadratic polynomial results to which the quadratic equation could be applied in order to determine the remaining roots.

$$P(x) = x^3 - 8x^2 + 24x - 32 = 0 \qquad \text{root: } x_1 = 4$$
$$\frac{x^3 - 8x^2 + 24x - 32}{x - 4} = (x^3 - 8x^2 + 24x - 32) : (x - 4) = \ldots$$

Division using polynomials is performed in exactly the same way as long division of whole numbers and can be solved by applying the same method:

$$
\begin{array}{l}
\big(\quad x^3 - 8x^2 + 24x - 32\big) : \big(x - 4\big) = x^2 - 4x + 8 \\
\underline{\;- x^3 + 4x^2\quad} \\
\qquad\; - 4x^2 + 24x \\
\qquad\;\; \underline{4x^2 - 16x} \\
\qquad\qquad\qquad 8x - 32 \\
\qquad\qquad\quad \underline{- 8x + 32} \\
\qquad\qquad\qquad\qquad 0
\end{array}
$$

2.4. Trigonometric Functions



Satz 2.1. *The sum of the angles in a triangle equals 180°*
Satz 2.2. *In a right triangle with sides a and b and hypotenuse c, the trigonometric functions for angle α opposite side a are:*

$$\sin\alpha = \frac{a}{c} \qquad \cos\alpha = \frac{b}{c} \qquad \tan\alpha = \frac{a}{b} = \frac{\sin\alpha}{\cos\alpha}$$

Satz 2.3. *(Pythagorean theorem) In a right triangle with sides a and b and hypotenuse c:*

$$a^2 + b^2 = c^2$$

From the Pythagorean theorem it follows directly that:
Satz 2.4. *The sum of the squares of the sine and cosine of the same angle equal one:*

$$\sin^2\alpha + \cos^2\alpha = 1$$

Proof.

$$a^2 + b^2 = c^2 \Leftrightarrow \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2} \Leftrightarrow \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \left(\frac{c}{c}\right)^2 \Leftrightarrow \sin^2\alpha + \cos^2\alpha = 1$$

□

Out of reasons of symmetry, it further follows that: $\cos(90° - \alpha) = \sin\alpha$ und $\sin(90° - \alpha) = \cos\alpha$

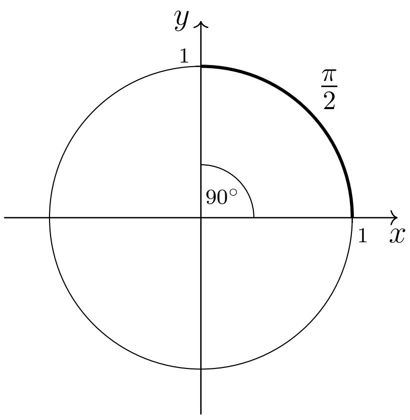
Radian measure

Every angle α can be unambiguously described by the length of the corresponding arc of the unit circle. The length of this arc is given in radians. The circumference of the unit circle is 2π.

It follows then that the arc of length π corresponds to an angle of 180° measured in degrees.

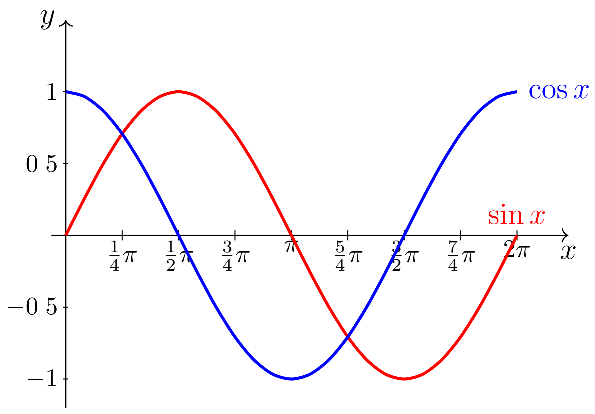
$$1^\circ \mathrel{\hat{=}} \frac{\pi}{180}$$

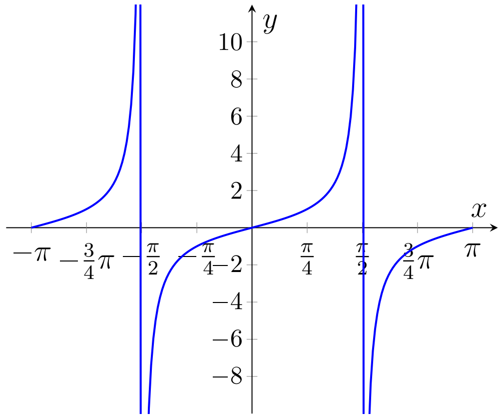
An angle α in degrees is converted into radians using : $rad(\alpha) = \alpha\frac{\pi}{180^\circ}$



The graphs of trigonometric functions

One should be familiar with the graphs of the trigonometric functions. The following diagram shows the respective excerpt over one period for the graphs of the sine and cosine functions as well as the tangent:





Important function values

Simple function values exist for trigonometric functions for 0° and for all multiples of 30° and 45°. It is sufficient to know these values for the corresponding values up to 90°, the values then repeat themselves with alternating algebraic sign according to the periodicity of the trigonometric function

.

Angle	Radians	Sine	Cosine	Tangent
0°	0	0	1	0
30°	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$
45°	$\frac{1}{4}\pi$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{1}{3}\pi$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{1}{2}\pi$	1	0	$\pm\infty$

As a memory aid: Half the square root of the whole numbers 0 to 4 provides the function values of the sine function for the angles 0°, 30°, 45°, 60°, 90°:

$$\frac{1}{2}\sqrt{0},\ \frac{1}{2}\sqrt{1},\ \frac{1}{2}\sqrt{2},\ \frac{1}{2}\sqrt{3},\ \frac{1}{2}\sqrt{4}$$

The order of the corresponding function values for the cosine function is the opposite. The function value for the tangent function can then be calculated from: $\tan x = \dfrac{\sin x}{\cos x}$

Periodicity and zeros

The periods of sine and cosine are each 2π, that of the tangent is π:

- $\sin(x + k \cdot 2\pi) = \sin(x)$ where $k \in \mathbb{Z}$
- $\cos(x + k \cdot 2\pi) = \cos(x)$ where $k \in \mathbb{Z}$
- $\tan(x + k \cdot \pi) = \tan(x)$ where $k \in \mathbb{Z}$

The zeros of the trigonometric functions are related as follows:

- $\sin x = 0 \Leftrightarrow \tan x = 0 \Leftrightarrow x = k\pi$ where $k \in \mathbb{Z}$
- $\cos x = 0 \Leftrightarrow x = k\pi + \frac{\pi}{2}$ where $k \in \mathbb{Z}$

2.5. The Inverse Function

If applying the chain rule to two functions (independent of the order of the functions) results in the **identity function** $h(x) = x$, then one function is the **inverse function** of the other. (For an explanation of the chain rule see also 2.10.)

$$(g \circ f)(x) = x = (f \circ g)(x)$$

The following example provides such a relationship:

$$f(x) = x + 1,\ g(x) = x - 1$$

$$(f \circ g)(x) = f(g(x)) = ((x - 1) + 1) = x$$
$$(g \circ f)(x) = g(f(x)) = ((x + 1) - 1) = x$$

Therefore f is the inverse function of g and vice versa. To denote the inverse function of a function f, the following notation is used:

$$f^{-1}(x)$$

CAREFUL:

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

The notation "raised to the -1" has its origins in linear algebra (matrices as maps) and is not subject to the rules for powers of real numbers discussed here.

It may be necessary to adjust the defined domain of a function in order to be able to denote the inverse function (see root functions). To find the inverse function, the function equation with form $y = f(x)$ is solved for x. The variable of the inverse function is subsequently often renamed x. The graph of the inverse function can be constructed from the graph of the original function by reflecting the graph across the identity function $f(x) = x$; this is the line through the origin with slope 1.

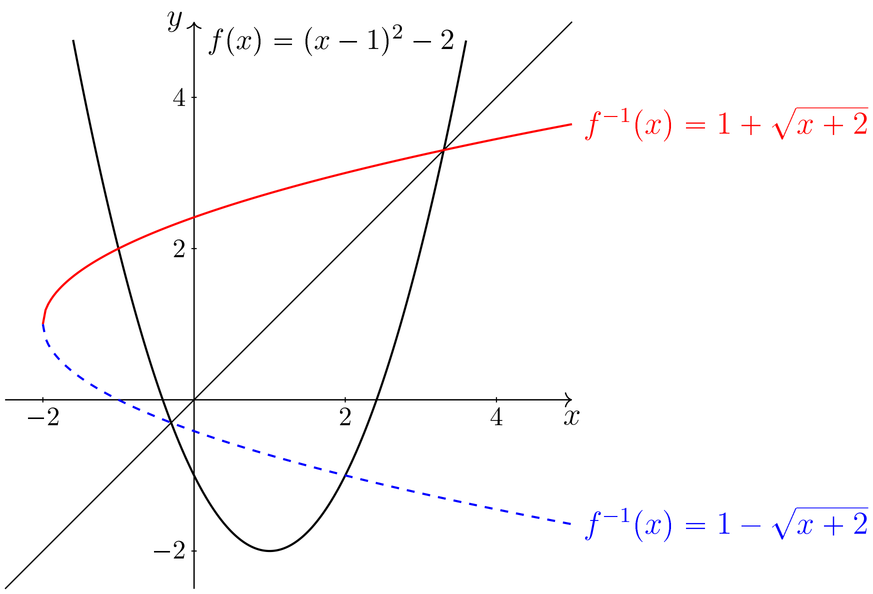
Note: You use the inverse function when you solve equations. In doing so, the inverse function is applied to both sides of the equation in order to solve for the variable. The example for this with which you may be most familiar might be the root function.

Examples:

1. $y = f(x) = \dfrac{1}{x+1} \Rightarrow x = \dfrac{1}{y} - 1 \Rightarrow f^{-1}(x) = \dfrac{1}{x} - 1$
2. $y = f(x) = e^{x+3} \Rightarrow \ln y = x + 3 \Rightarrow x = \ln y - 3 \Rightarrow f^{-1}(x) = \ln x - 3$
3. $y = f(x) = (x-1)^2 - 2 \Rightarrow x - 1 = \pm\sqrt{y+2} \Rightarrow x = 1 \pm \sqrt{y+2}$

$$\Rightarrow \begin{cases} f^{-1}(x) = 1 + \sqrt{x+2}, & f^{-1}(x) \geqslant 1 \\ f^{-1}(x) = 1 - \sqrt{x+2}, & f^{-1}(x) < 1 \end{cases}$$

   Therefore $f^{-1}(x) = 1 + \sqrt{x+2}$ is the inverse function of
   $f : [1, \infty) \to \mathbb{R}$ where $f(x) = (x-1)^2 - 2$
   and $f^{-1}(x) = 1 - \sqrt{x+2}$ is the inverse function of
   $f : (1, -\infty) \to \mathbb{R}$ where $f(x) = (x-1)^2 - 2$. The following diagram shows the parabola given by f as well as the graphs of both of the inverse functions:



2.6. Root Functions

The root function $f^{-1}(x) = \sqrt{x} = x^{\frac{1}{2}}$ is the inverse function of $f(x) = x^2$ for $\mathbf{x} \geqslant \mathbf{0}$.

Examples:

1. $x^2 = 4 \Leftrightarrow \pm\sqrt{x^2} = \pm\sqrt{4} \Leftrightarrow x = \pm 2$
2. $27x^3 = 8 \Leftrightarrow \sqrt[3]{27x^3} = \sqrt[3]{8} \Leftrightarrow 3x = 2 \Leftrightarrow x = \frac{2}{3}$
3. $\dfrac{x^4}{5} = 2000 \Leftrightarrow x^4 = 10000 \Leftrightarrow \pm\sqrt[4]{x^4} = \pm\sqrt[4]{10000} \Leftrightarrow x = \pm 10$

2.7. Exponential Functions and Logarithms

An exponential function is a function with the following mapping rule:

$$f(x) = a^x,\ a \in \mathbb{R},\ const$$

The “const“ label denotes that a is a constant. The base e exponential function (sometimes called Euler’ number) is known as the **natural exponential function** and has numerous applications in many areas of engineering and economic mathematics.

$$f(x) = e^x$$

Its inverse function is the natural logarithm: $f^{-1}(x) = \ln x$

References to“the exponential function“ generally mean the natural exponential function. References to “the logarithm function“ or “the logarithm“ generally mean the natural logarithm function.

Applying the arithmetic rules for exponentials and the rules for logarithm arithmetic derived from them, one easily arrives at the following relationships:

$$a^x = e^{x \ln a},\ a \in \mathbb{R}^+ \text{ and } \log_b x = \frac{\ln x}{\ln b},\ b \in \mathbb{R}^+$$

2.8. Symmetry Relations of Functions

Definition 2.5:
A function f is called even if $f(-x) = f(x)$ applies. The function is called odd if the condition $f(-x) = -f(x)$ is met.

Examples:

1. $\cos x = \cos(-x)$, $x^2 = (-x)^2$
2. $-\tan x = \tan(-x)$, $x^3 = -(-x)^3$

The graph of an even function is symmetric to the y-axis. If a function is odd, its graph is point-symmetric to the origin..

2.9. Translation, Stretching and Compression

A constant $c \in \mathbb{R}$ can be used to create transformations of the graph of a function:

$$f(x+c) \quad \Rightarrow \quad \begin{cases} c > 0 & \text{Translation by } c \text{ to the left} \\ c < 0 & \text{Translation by } |c| \text{ to the right} \end{cases}$$

$$f(x) + c \quad \Rightarrow \quad \begin{cases} c > 0 & \text{Translation by } c \text{ upwards} \\ c < 0 & \text{Translation by } |c| \text{ downwards} \end{cases}$$

$$f(cx) \quad \Rightarrow \quad \begin{cases} & \text{horizontal compression by a factor} |c| \\ c < 0 & \text{in addition: reflection about the vertical axis} \end{cases}$$

$$cf(x) \quad \Rightarrow \quad \begin{cases} & \text{vertical stretching by a factor } |c| \\ c < 0 & \text{in addition: reflection about the horizontal axis} \end{cases}$$

Examples:

1. $\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$ The graph of the cosine function corresponds to that of the sine function transferred left by $\frac{\pi}{2}$.
2. $2\sin x \Leftrightarrow$ Doubling of the amplitude of $\sin x$
3. $\sin(2x) \Leftrightarrow$ Doubling of the frequency of $\sin x$

2.10. Combination and Composition of Functions

The addition, multiplication, etc. of functions is called (algebraic) combination. These produce new functions whose domain is the intersection of the preceding domains. In the case of division, it is also necessary to ensure that the denominator does not become zero.

Examples:

Applying $f(x) = x + 1$, $g(x) = x - 1$, $D_f = D_g = \mathbb{R}$ it is possible to generate the following new functions:

1. $f(x)g(x) = (x+1)(x-1) = x^2 - 1 = r(x)$, $D = D_f \cap D_g = \mathbb{R}$
2. $h(x) = \dfrac{f(x)}{g(x)} = \dfrac{x+1}{x-1}$ In this case the domain has changed: $D = \mathbb{R}\backslash\{1\}$

A composition is the point wise application of one function to the result of another to produce a third function:

$$(f \circ g)(x) = f(g(x))$$

Note: In general, composition is **not commutativ**e: $(f \circ g)(x) \neq (g \circ f)(x)$

Examples:

With $f(x) = \frac{1+x}{1-x}$, $g(x) = \frac{1}{1+x}$ where $x \neq 1$ and $x \neq -1$, the following compositions are possible:

1. $(f \circ g)(x) = f(g(x)) = \frac{1+g(x)}{1-g(x)} = \frac{1+\frac{1}{1+x}}{1-\frac{1}{1+x}} = \frac{\frac{1+x}{1+x}+\frac{1}{1+x}}{\frac{1+x}{1+x}-\frac{1}{1+x}} = \frac{2+x}{x} = 1+\frac{2}{x}$
   Regarding the domain: Since g is only defined for $x \neq -1$ and $g(0) = 1$ presents the critical point for f, the domain is $D = \mathbb{R} \setminus \{-1{,}0\}$
2. Analogously, it follows that
   $(g \circ f)(x) = g(f(x)) = \frac{1}{1+f(x)} = \frac{1}{1+\frac{1+x}{1-x}} = \frac{1}{2}(1-x)$
   Regarding the domain: Since g requires only $x \neq -1$ and $f(x) \neq -1$ for all $x \in \mathbb{R}$, the domain is $D = \mathbb{R} \setminus \{1\}$

Combining with the absolute value function:

- $|f(x)| \Rightarrow$ Reflection of negative function values about the horizontal axis
- $f(|x|) \Rightarrow$ Reflection of the function values for positive values of x about the vertical axis

3. Differential and Integral Calculus

NOTE: In the following you will find strictly a short application-oriented summary of the simplest aspects of differential and integral calculus. Important details and exact formulations are not included. Both of these will be provided within the framework of the math lectures. However, you should possess a rudimentary understanding of differential and integral calculus before you even begin your studies because this knowledge will be expected in other lectures.

3.1. Differential Calculus

Basic derivatives and the differentiation rules

Definition 3.1:
The derivative of function f of a variable x is designated, among other things, by$f'(x)$.

$f(x)$	ax	x^n	$\ln x$	$\frac{1}{x}$	$\sqrt{x}$	$-\cos x$	$\sin x$	e^x	$\arctan x$
$f'(x)$	$a = const.$	nx^{n-1}	$\frac{1}{x}$	$\frac{-1}{x^2}$	$\frac{1}{2\sqrt{x}}$	$\sin x$	$\cos x$	e^x	$\frac{1}{1+x^2}$

Satz 3.1. *If $f, g : D \subseteq \mathbb{R} \to \mathbb{R}$ are differentiable functions and $c \in \mathbb{R}$ is an arbitrary constant, then:*

Linearity	$(cf)' = c \cdot f'$	$(cf(x))' = c \cdot f'(x)$
Linearity	$(f+g)' = f' + g'$	$(f(x) + g(x))' = f'(x) + g'(x)$
Product rule	$(f \cdot g)' = f'g + fg'$	$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$
Quotient rule	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$	$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
Chain rule	$(f \circ g)' = f' \cdot g'$	$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

Tip: You should memorize these basic derivatives and the differentiation rules. They can be applied to determine every other derivation.

Examples:

1. The derivatives of functions with fractions or roots can be determined by transforming them into exponential expressions and subsequently applying the differentiation rule for a monomial:
   $(\sqrt{x})' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$
2. $\left(\frac{1}{\sqrt[3]{x}}\right)' = \left(x^{-\frac{1}{3}}\right)' = -\frac{1}{3}x^{-\frac{4}{3}} = -\frac{1}{3 \cdot \sqrt[3]{x^4}}$
3. $(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \begin{cases} \frac{1}{\cos^2 x} \text{or} \\ 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x \end{cases}$

4. $(3\sin x)' = 3(\sin x)' = 3\cos x$
5. $(\sin x \pm \ln x)' = \cos x \pm \frac{1}{x}$
6. $(\sin x \cdot \ln x)' = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$
7. $\left(\frac{\sin x}{\ln x}\right)' = \dfrac{\cos x \cdot \ln x - \sin x \cdot \frac{1}{x}}{\ln^2 x}$
8. $(\sin(\ln x))' = \cos(\ln x) \cdot \frac{1}{x}$
9. $(\ln(\sin x))' = \dfrac{1}{\sin x} \cdot \cos x = \dfrac{1}{\tan x} = \cot x$

3.2. Application of Differential Calculus

Local rates of change

In many applications, the local rate of change of a function is of interest. “Local” means at a point $x_0 \in D$. This local rate of change is the slope of the function at point x_0. It reveals by approximately how much the value of the function will change for a small change in x_0.

Example:

The local rate of change of function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ at point $x_1 = 3$ is given by the value of the function of the derivative f' at point$x_1 = 3$:

$$f'(x) = 2x \Rightarrow f'(3) = 2 \cdot 3 = 6$$

The slope of the function f at point $x_1 = 3$ is 6. That means that the value of the function changes by approximately 6 times the change in the x-value if the latter is changed by only a slight amount.

Tangent to the plot of a function

The equation of a tangent at point $x_0 \in D$ on the plot of function f is obtained by solving for the ratio of the vertical change to the horizontal change (rise over run). (Compare: point-slope equation for a straight line)

Example:

The objective is to solve for the tangent at point $x_0 = 1$ on the plot of function $f = x^2 - 4$.

The point through which the tangent passes is determined by the given point and the value of the function at that point:

$$P = (1, f(1)) = (1, -3)$$

The slope is equal to the derivative of f at that point:

$$f'(x_0) = f'(1) = 2 \cdot 1 = 2$$

So solving for the rise over run for the equation of the function for tangent t:

$$2 = \frac{t(x) - (-3)}{x - 1} \Leftrightarrow t(x) + 3 = 2x - 2 \Leftrightarrow t(x) = 2x - 5$$

Extreme values

Definition 3.2:
Extreme values (also: Extreme points) are the points $x \in D$ at which the function assumes a local minimum or maximum.

ATTENTION 1: A point $x \in D$ itself is **not** a minimum or maximum!

ATTENTION 2: Points of inflection are not extreme points!

Satz 3.2. *If a function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ in $x_0 \in (a, b)$ has a* ***local minimum or maximum*** *and is differentiable atx_0, then*

$$f'(x_0) = 0$$

A **necessary** condition for an extreme point is therefore that the first derivative at that point is zero! This condition is used to find possible candidates for extreme points. The candidates are subsequently individually checked for their properties.

In order to determine whether a function is rising or falling around at extreme point x_0, the second derivative of the function at that point is considered.

Satz 3.3. *If a function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ in $x_0 \in (a, b)$ can x_0 can be differentiated twice and $f'(x_0) = 0$ then:*

$$f \text{ has a local } \textbf{minimum} \text{ at } x_0 \text{ if } f''(x_0) > 0$$

$$f \text{ has a local } \textbf{maximum} \text{ at } x_0 \text{ if } f''(x_0) < 0$$

Notes

1. Where applicable, the **edges** of a closed interval D must be investigated individually.
2. If $f''(x_0) = 0$, this theorem is not conclusive. The point must then be investigated by other means.

Example:

In order to investigate the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \frac{1}{3}x^3 - 4x^2 + 7x - 1$ for extreme points, the required derivatives are first determined:

$$f'(x) = x^2 - 8x + 7 \text{ and } f''(x) = 2x - 8$$

The disappearing first derivative provides the candidates for extreme points:

$$f'(x) = 0 \Leftrightarrow x^2 - 8x + 7 = 0 \Leftrightarrow x_1 = 1, \; x_2 = 7$$

The second derivative provides the criteria on whether and which extreme points exist:

$$f''(x_1) = -6 < 0 \Rightarrow f \text{ has a local maximum } x_1 = 1$$
$$f''(x_2) = 6 > 0 \Rightarrow f \text{ has a local minimum at } x_2 = 7$$

3.3. Integral Calculus

Definite integrals are integrals with integration limits. A simple interpretation of an integral is that its the value of its sums corresponds to the area between the curve and the x-axis;

whereby areas under the x-axis must be subtracted. Definite integrals are solved by applying the following theorem:

Satz 3.4. *Function* $F : [a, b] \to \mathbb{R}$ *is continuously differentiable over* (a, b) *and its derivative* F' *can be integrated, then:*

$$\int_a^b F'(x)dx = F(b) - F(a) =: F|_a^b =: [F]_a^b$$

That means that in order to calculate an integral $\int_a^b F'(x)dx$ whose integrand is the derivative of F, it is merely necessary to insert the integration limits in F and to determine the difference $F(b) - F(a)$. In general, function F is not known and must be determined from the integrand:

$$\int_a^b f(x)dx = \int_a^b F'(x)dx$$

Definition 3.3:
If the derivative of function F is function f, then F is called ***an* anti-derivative** of f.

ATTENTION: F is ***not* the** anti-derivative of f; "the" anti-derivative does not exist. Two different anti-derivatives of a function differ by a constant.

Important anti-derivatives

The following table lists an anti-derivative F for each function f. It is possible to add a constant $c \in \mathbb{R}$ to each of these anti-derivatives.

$f(x)$	1	x	$x^n, n \neq -1$	$\frac{1}{x}$	$\frac{1}{x^2}$	$\sin x$	$\cos x$	e^x	$\frac{1}{1+x^2}$	$\sqrt{x}$
$F(x)$	x	$\frac{1}{2}x^2$	$\frac{1}{n+1}x^{n+1}$	$\ln\lvert x\rvert$	$-\frac{1}{x}$	$\cos x$	$\sin x$	e^x	$\arctan x$	$\frac{2}{3}x^{\frac{3}{2}}$

The rules for exponential arithmetic also allow all other root expressions and fractions to be simply integrated by applying the integration rule for the monomial. Many integrals can be solved by applying the **linearity of integration**:

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx \text{ and } \int c \cdot f(x)dx = c \int f(x)dx,\ c \in \mathbb{R}$$

Examples:

1. Applying linearity and the anti-derivative for x^n allows integrating polynomials+:
   $\int_{-1}^2 (x^3 + 2x - 1)\ dx = \left[\frac{1}{4}x^4 + x^2 - x\right]_{-1}^2 = \left(\frac{16}{4} + 4 - 2\right) - \left(\frac{1}{4} + 1 + 1\right) = \frac{15}{4}$
2. Determination of the surface area beneath the sinusoidal between 0 and π:
   $\int_0^\pi \sin(x)dx = [-\cos(x)]_0^\pi = -\cos(\pi) - (-\cos(0)) = 2$

4. Vector Algebra

4.1. Geometry Equations

In the following, a is always the base of the observed surface. The height is determined orthogonally to the base.

Rectangle

For a rectangle with sides of lengths a and b the following applies:

- Perimeter: $U = 2 \cdot a + 2 \cdot b$
- Area: $A = a \cdot b$

Parallelogram

For a parallelogram with sides of lengths a and b and height h the following applies:

- Perimeter: $U = 2 \cdot a + 2 \cdot b$
- Area: $A = a \cdot h$

Circle

For a circle with radius r the following applies:

- Circumference: $U = 2 \cdot \pi \cdot r$
- Area: $A = \pi \cdot r^2$

Triangle

For a triangle with sides of length a, b and c and height h the following applies:

- Perimeter: $U = a + b + c$
- Area: $A = \frac{1}{2} \cdot a \cdot h$

Cuboid

For a cuboid with sides of length a, b and c the following applies:

- Surface area: $O = 2 \cdot a \cdot b + 2 \cdot b \cdot c + 2 \cdot a \cdot c$
- Volume: $V = a \cdot b \cdot c$

Pyramid (with a square base)

For a pyramid with base side length a and height h the following applies:

- Surface area: $O = a^2 + 2 \cdot a \cdot \sqrt{h^2 + (\frac{a}{2})^2}$
- Volume: $V = \frac{1}{3} \cdot a^2 \cdot h$

Cone

For a cone with radius r and height h the following applies:

- Surface area: $O = \pi \cdot r^2 + \pi \cdot r \cdot \sqrt{h^2 + r^2}$
- Volume: $V = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$

Cylinder

For a cylinder with radius r and height h the following applies:

- Surface area: $O = 2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h$
- Volume: $V = \pi \cdot r^2 \cdot h$

Sphere

For a sphere with radius r the following applies:

- Surface are: $O = 4 \cdot \pi \cdot r^2$
- Volume: $V = \frac{4}{3} \cdot \pi \cdot r^3$

4.2. Vectors

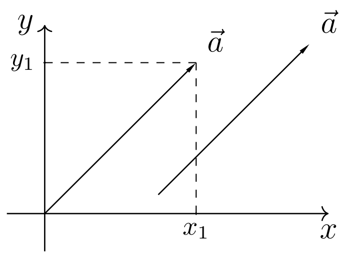
Definition 4.1:
Elements whose values are real numbers are called **scalars**. In contrast to this, elements whose complete characterization requires **a magnitude as well as a direction** (and sometimes a direction of rotation) are called **vectors**. (I. N. Bronstein, K. A. Semendjajew, G. Musiol, and H. Mühlig. Taschenbuch der Mathematik. Verlag Harry Deutsch, 2006.)

- Examples of scalars include: Mass, temperature, energy, and work
- Examples of vectors include: Force, velocity, acceleration, angular frequency, as well as electric and magnetic field strength.

The notation for vectors depends on the application. Typically, lower case letters are used, usually in bold face and with a small arrow:

$$\vec{a} \text{ or } \mathbf{a}$$

The requirement that a vector should have an **explicit direction** means that for concrete vectors, it is first necessary to define **in which space** they are located. The characteristics length and direction can then be represented by arrows in the respective space. These arrows can point in different respective directions and have different respective lengths. One direct conclusion from the definition is that **all vectors that are parallel and offset are identical** since they have the same direction and length.



Further, it follows that all vectors can be placed with their initial point at the origin (this special representation is also known as the **position vector**). It then follows from this that the point at which the tip of the vector lies exactly defines the vector. For this reason, vectors can be represented by a tuple of coordinates just as points can:

Definition 4.2:
An n-dimensional vector is an n-tuple of real numbers:

$$\vec{a} = \begin{pmatrix} a_1 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}, \ a_i \in \mathbb{R}$$

The a_i are called coordinates or components of $\vec{a}$.

4.2.1. Mathematical Operations with Vectors

For vectors$\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$ and scalars $\lambda, \mu \in \mathbb{R}$ the following applies:

1. Vector addition: $\vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$
2. Multiplication with a scalar: $\lambda\vec{a} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{pmatrix}$

	Vector addition	Multiplication with a scalar
Commutative law	$\vec{a} + \vec{b} = \vec{b} + \vec{a}$	$\lambda\vec{a} = \vec{a}\lambda$
Associative law	$\vec{a} + \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) + \vec{c}$	$(\lambda\mu)\,\vec{a} = \lambda\,(\mu\vec{a})$
Example of a neutral element	$\vec{a} + \vec{0} = \vec{a}$	$\vec{a} \cdot 1 = \vec{a}$
Example of inverse element	$\vec{a} + (-\vec{a}) = \vec{0}$	$\lambda\vec{a} = \vec{b} \Leftrightarrow \vec{a} = \frac{1}{\lambda}\vec{b}$
Distributive law	$\lambda\left(\vec{a} + \vec{b}\right) = \lambda\vec{a} + \lambda\vec{b}$ and $(\lambda + \mu)\,\vec{a} = \lambda\vec{a} + \mu\vec{a}$	

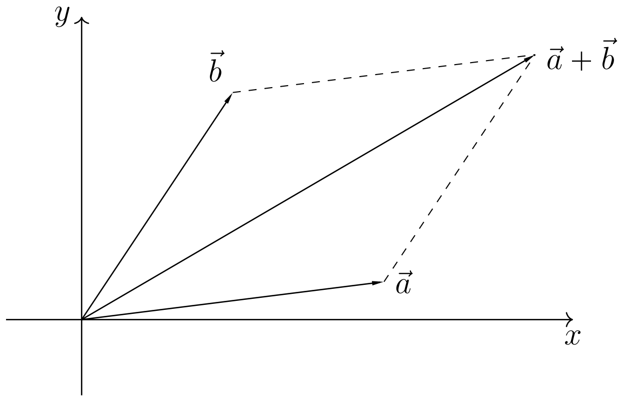
Note: Subtraction follows from 1. and 2. above as addition with the vector multiplied by -1: $\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = \vec{a} + ((-1)\vec{b})$

Examples:

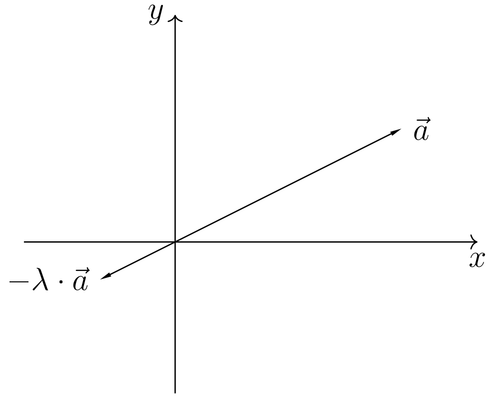
$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}, \quad 5 \cdot \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$$

Geometric interpretation (only meaningful for $\mathbb{R}^2$ and $\mathbb{R}^3$):

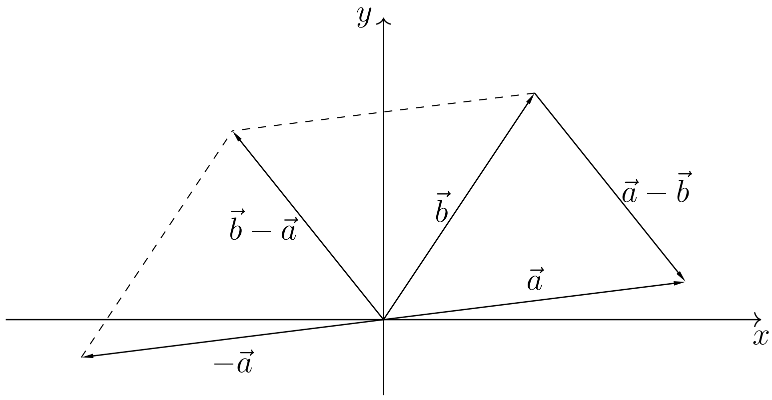
- The addition of two vectors is possible through parallel offset:



- The multiplication with a scalar λ corresponds to a stretching by λ. If $\lambda < 0$, this multiplication also corresponds to a reflection about the origin (here with $|\lambda| < 1$):



- Subtraction follows from addition with a vector multiplied by -1. This also makes clear that the vector $\vec{a}-\vec{b}$ results from the vector $\vec{b}-\vec{a}$ by inverting the direction.



Magnitude of a vector

In the case of Cartesian coordinate representation in a plane, the magnitude (also called length or modulus) of a vector is determined using the Pythagorean theorem:

$$|\vec{v}| = \sqrt{(v_1)^2 + (v_2)^2}$$

This definition applies analogously to any higher-dimensioned vector space of arbitrary dimension n:

$$|\vec{v}| = \sqrt{(v_1)^2 + (v_2)^2 + \ldots + (v_n)^2}$$

Distance between two vectors

The distance between two vectors is then given by the length of the vector which is the difference between the two vectors. This is equivalent to the distance between the two points which are represented by the tuple.

$$d\left(\vec{a},\vec{b}\right) = \left|\vec{a}-\vec{b}\right|$$

Example

Confirm by recalculating that vector $\vec{b}$ is twice as long as vector $\vec{a}$ when the following applies: $2\vec{a} = \vec{b}$ and $\vec{a},\vec{b} \in \mathbb{R}^2$

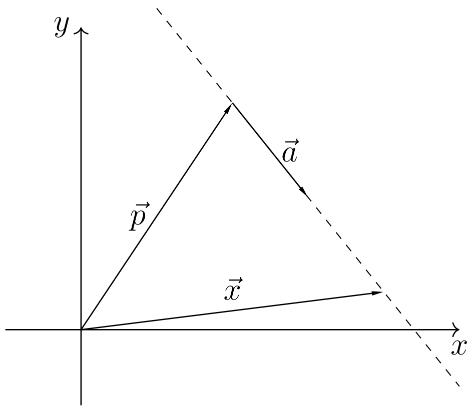
The vector form of the equation for a line:

$$\vec{x} = \vec{p} + \lambda\vec{a},\ \lambda \in \mathbb{R}$$

Where:

- $\vec{p}$ is the position vector (The starting point)
- $\vec{a}$ is the direction vector, this may not be the zero vector
- λ is the parameter

By starting from $\vec{p}$,every point $\vec{x}$ on the line can be determined by adding $\lambda\vec{a}$.



Consequences:

- A point Q given by its position vector $\vec{q}$ lies on the line $g:\ \vec{x} = \vec{p} + \lambda\vec{a}$, if there is a λ for which $\vec{q} = \vec{p} + \lambda\vec{a}$ applies.
- Two lines are parallel to each other if their direction vectors are collinear: $g_1 \| g_2 \Leftrightarrow \vec{a_1} \| \vec{a_2}$
- Two lines are identical if they are parallel and (at least) one point exists which satisfies both equations for the lines. $g_1 = g_2 \Leftrightarrow g_1 \| g_2 \wedge \vec{p_1} + \lambda\vec{a_1} = \vec{q} = \vec{p_2} + \lambda\vec{a_2}$

Examples:

- $g:\ \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow P = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \in g,\ Q = \begin{pmatrix} 5 \\ 6 \\ 3 \end{pmatrix} \notin g$
- $g_1:\ \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ g_2:\ \vec{x} = \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} a \\ 2 \\ 2 \end{pmatrix} \Rightarrow g_1 \| g_2$ for $a = 2$, otherwise :

$g_1 \nparallel g_2$

- The lines$g_1 : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $g_2 : \vec{x} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ are identical because they are parallel $g_1 \| g_2$ and the point $P = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, which definitely lies on g_1 also lies on g_2 when $\mu = -1$.

The intersection of two lines

The **point of intersection** (given by $\vec{q}$) satisfies both equations for the lines:

$$g_1 : \vec{p} + \lambda \vec{a} = \vec{q} = \vec{r} + \mu \vec{b} : g_2$$

This provides a system of equations: Each component of $\vec{q}$ provides an equation with the unknown variables λ, μ. If solutions $\lambda, \mu \in \mathbb{R}$ can be found such that all equations are satisfied, the intersection of the lines can be determined from either of the two equations using these solutions.

Example:

The point of intersection S of the two line:

$$g_1 : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \, g_2 : \vec{x} = \begin{pmatrix} 0 \\ 7 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

shall be determined, in the event that the two lines intersect.
The following must hold:

$$\vec{s} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

This leads to the system of equations:

$$\begin{aligned} 1 + \lambda &= \mu \\ 2 + \lambda &= 7 - 2\mu \\ 3 + \lambda &= 6 - \mu \end{aligned}$$

Adding equations (1) and (3) results in: $4 + 2\lambda = 6 \Rightarrow \lambda = 1$, leading to $\mu = 2$ (for example by using Equation (1)). Since this is an over-determined system, the validity of the solution must be verified using the other equations. In this case it is also possible to determine the intersection with one line equation and check it with another:

$$\vec{s} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 6 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

The Scalar Product

There are various possibilities to define the products of vectors. One of these is the scalar product, which results in a scalar.

Definition 4.3:
The scalar product of two vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$ is given by:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots a_n b_n$$

Example:

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} 4 \\ -1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} = 8 - 2 + 12 = 18$$

Properties of the scalar product

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
2. $\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
3. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
4. $\lambda \left(\vec{a} \cdot \vec{b}\right) = \vec{a} \cdot \lambda\vec{b} = \lambda\vec{a} \cdot \vec{b}$
5. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\alpha \,\text{where}\, \alpha = \sphericalangle \left(\vec{a}, \vec{b}\right)$ This also leads to: $\alpha = \arccos\left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)$
   In addition, it follows that two vectors are exactly perpendicular to one another when $\vec{a} \cdot \vec{b} = 0$.
6. $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$

The Vector Product or Cross Product

Another possibility to define the products of vectors is the vector product for vectors in $\mathbb{R}^3$. The result is a vector. The vector product is not a component of prerequisite school math but may find application in lectures before it is addressed in the mathematics lecture.

Definition 4.4:
The vector product of two vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$ is given by:

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Note:

1. Schematically the calculation results in three crosses:

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \times\, bottom \\ - \times\; large \\ \times\, top \end{pmatrix}$$

Where:

$$\times\, bottom = \begin{matrix} a_2 & & b_2 \\ & \times & \\ a_3 & & b_3 \end{matrix} = a_2b_3 - a_3b_2$$

$$-\times\ large = (-1) \cdot \begin{matrix} a_1 & & b_1 \\ & \times & \\ a_3 & & b_3 \end{matrix} = -(a_1b_3 - a_3b_1) = a_3b_1 - a_1b_3$$

$$\times\, top = \begin{matrix} a_1 & & b_1 \\ & \times & \\ a_2 & & b_2 \end{matrix} = a_1b_2 - a_2b_1$$

Example:

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

Properties of the cross product

1. $\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$ (conditionally commutative)
2. $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (distributive law)
3. $\lambda\left(\vec{a} \times \vec{b}\right) = \lambda\vec{a} \times \vec{b} = \vec{a} \times \lambda\vec{b}$
4. The (double) vector product is generally not associative: $\vec{a} \times \left(\vec{b} \times \vec{c}\right) \neq \left(\vec{a} \times \vec{b}\right) \times \vec{c}$
5. The outcome of the cross product $\vec{a} \times \vec{b}$ is a vector $\vec{c}$ which is orthogonal to $\vec{a}$ and to $\vec{b}$: $\vec{a} \times \vec{b} \perp \vec{a}$ and $\vec{a} \times \vec{b} \perp \vec{b}$, and therefore $\vec{a} \times \vec{b} \perp E\left(\vec{a}, \vec{b}\right)$ as well, wherein $E\left(\vec{a}, \vec{b}\right)$ describes the plane defined by $\vec{a}$ and $\vec{b}$.
6. $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\alpha$ where $\alpha = \measuredangle\left(\vec{a}, \vec{b}\right)$ Note: $\alpha \in [0, \pi] \Rightarrow \sin\alpha \geqslant 0$
7. This leads directly to: The length $\left|\vec{a} \times \vec{b}\right|$ is equal to the area of the parallelogram whose adjacent sides are the vectors $\vec{a}$ and $\vec{b}$.
8. If $\vec{a}, \vec{b}$ are collinear, then $\vec{a} \times \vec{b} = \vec{0}$. The cross product disappear exactly then, when the vectors $\vec{a}$ and $\vec{b}$ are linearly dependent. $\vec{a} \times \vec{a} = 0$ applies in particular.
9. $\left|\vec{a} \times \vec{b}\right|^2 = |\vec{a}|^2|\vec{b}|^2 - \left(\vec{a} \cdot \vec{b}\right)^2$

5. Stochastic Processes

5.1. Relative and Absolute Frequencies

Various characteristic attributes emerge when data is evaluated. During the evaluation process, sample size n represents the number of data sets. The **absolute frequency** H_i represents the number of times the characteristic attribute a_i appears and the **relative frequency** h_i represents that portion which the characteristic attribute a_i has in n .

For example, the chosen degree course of all new students can be recorded as a characteristic attribute. The number of all students beginning their degree and thus the sample size is 300. Thus, the absolute frequencies of the characteristic attributes is given by the number of mechanical engineers, computer scientists, etc. A total of 50 of the 300 beginning students are mechanical engineers, that means $H_i = 50$ and

$$h_i = \frac{H_i}{n} = \frac{50}{300} = \frac{1}{6} \approx 0\ 17$$

If there are k characteristic attributes, then $0 \leqslant H_i \leqslant n$ applies for all absolute frequencies of the characteristic attributes and

$$H_1 + H_2 + \cdots + H_k = n.$$

and $0 \leqslant h_i \leqslant 1$ applies for the corresponding relative frequencies as well as

$$h_1 + h_2 + \cdots + h_k = 1.$$

In the event that the characteristic attributes are all numerical values, then the term **quantitative characteristic** is used and the **mean** $\bar{x}$ of this characteristic is calculated from the arithmetic average.

$$\begin{aligned} \bar{x} &= \tfrac{1}{n} \cdot (x_1 + x_2 + \cdots + x_n) \\ &= \tfrac{1}{n} \cdot (a_1 \cdot H_1 + a_2 \cdot H_2 + \cdots + a_k \cdot H_k) \\ &= a_1 \cdot h_1 + a_2 \cdot h_2 + \cdots + a_k \cdot H_k \end{aligned}$$

For the attribute "age", the mean is the average age of all beginning students.

Age	17	18	19	20	21	22	23	25
Frequency	2	78	113	48	18	14	15	12

In our case the average age is given by $\bar{x} = 19\ 59$ years.

A key characteristic parameter is given by the average squared deviation from the average value, known as the **variance** or **scatter**.

$$\sigma^2 = (x_1 - \bar{x})^2 \cdot h_1 + (x_2 - \bar{x})^2 \cdot h_2 + \cdots + (x_k - \bar{x})^2 \cdot h_k.$$

The squaring prevents positive and negative deviations from the average value from cancelling each other out. However, this means that the variance therefore does not have the same units as the average.

In the case of the average age of the beginning students, the variance is calculated as

$$\sigma^2 = (17 - 19\ 59)^2 \cdot \frac{2}{300} + \cdots + (25 - 19\ 59)^2 \cdot \frac{12}{300} = 3\ 0025$$

The **standard deviation**

$$\sigma = \sqrt{\sigma^2}$$

can be used as a measure for the scattering about the average value. It describes how good a representation the average value is for the data set. It always has the same units as the average and is on the scale of $\sigma \approx 1{,}73$,for example.

5.2. Probabilities

Whereas the relative frequencies provide an evaluation of the past, probabilities are used in order to establish a prognosis for the future. The probability therefore provides the relative frequency that can be approximately expected for a specific event.

A random experiment is an experiment with a random outcome which can be repeated ad infinitum under the same conditions. The set of all possible outcomes from a random experiment is called the **sample space**. An experiment for which all outcomes have the same probability is called a **Laplace experiment**. If only a single outcome (hit) and its complement (miss) are observed for an experiment, this is referred to as a **Bernoulli experiment**.

The probabilities of the outcomes e_i are designated by $P(e_i) = p_i$. The same rules which apply to the relative frequencies also apply to the probabilities of the outcomes, that is $0 \leqslant p_i \leqslant 1$ and $p_1 + p_2 + \cdots + p_k = 1$.

An **event** is a subset of the sample space. The probability $P(E)$ of an event E is determined by adding all the probabilities of the corresponding events. The complement$\bar{E}$ of E contains all outcomes which do not belong to E. $P(E) + P(\bar{E}) = 1$ applies. The probability of the complement can therefore be calculated based on the probability of the event and vice versa.

All events which simultaneously lie in the events E and F form the intersection $E \cap F$. All events which lie in E or F form the union $E \cup F$.

Here an example for this: One all is randomly drawn from an urn which contains 7 green balls and 2 orange balls. Using the abbreviations "o" for orange and "g" green, the probability of drawing a green ball is $P(g) = \frac{7}{9}$ and the probability of drawing an orange ball is $P(o) = \frac{2}{9}$. the event "an orange ball is drawn" corresponds to "no green ball is drawn" and therefore is the complement of the event "a green ball is drawn".

Now two balls are drawn randomly one after the other, for each the color is noted and the drawn ball is immediately returned to the urn. The resulting sample space is $S = \{gg,\ go,\ og,\ oo\}$. In our experiment with the urn we observe the events

E: "At least one ball is green" and F: "Exactly one ball is orange".

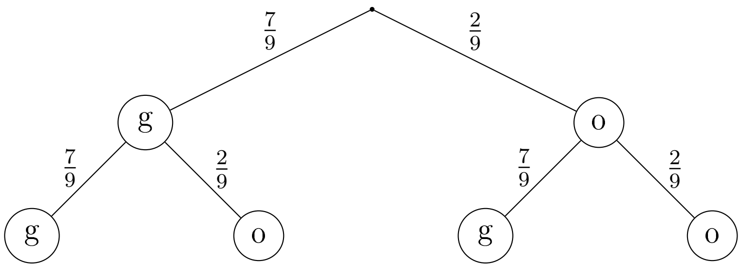
In set notation, this results in $E = \{gg,\ go,\ og\}$ and $F = \{go,\ og\}$.

As the intercept of the events one obtains $E \cap F = \{og,\ go\}$ and as the union one obtains $E \cup F = \{gg,\ go,\ og\}$.

Arithmetic with probabilities

Multi-step random experiments can be described by a tree diagram. Each event of the complete experiment corresponds to one path from the root of the tree to the tip of a branch.

The urn example described above is a two-step random experiment and the tree diagram is composed as follows.



Rule of product and rule of sum

The probabilities of outcomes and events of multi-step random experiments can be calculated especially easily through the help of a tree diagram.

Rule of product: The probability of an event is equal to the product of the probabilities along the length of the respective path.

The probability of drawing two green balls in our experiment is $P(gg) = \frac{7}{9} \cdot \frac{7}{9} = \frac{49}{81}$.

Rule of sum: The probability of an event is equal to the sum of the probabilities of the outcomes contained in that event.

Therefore the probability of drawing a minimum of one green ball is

$$P(E) = P(gg \vee go \vee og) = P(gg) + P(go) + P(og) = \frac{49}{81} + \frac{14}{81} + \frac{14}{81} = \frac{77}{81}.$$

Generalized rule of sum

For two events E and F, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ applies.

For the events E and F of the urn experiment described above

$P(F) = \frac{28}{81}$, $P(E) = \frac{77}{81}$, $P(E \cap F) = \frac{28}{81}$ and $P(E \cup F) = \frac{28}{81} + \frac{77}{81} - \frac{28}{81} = \frac{77}{81}$ applies.

Conditional probability, dependence of events

The probability that an event F occurs under the condition that the event E has occurred previously is called the **conditional probability of F given the condition E** and is designated by $P(F|E)$. The following applies:

$$P(F|E) = \frac{P(E \cap F)}{P(E)}.$$

Two events E and F are **independent** exactly when $P(E \cap F) = P(E) \cdot P(F)$ applies.

For our example, the conditional probability of F given E is:

$$P(F|E) = \frac{\frac{28}{81}}{\frac{77}{81}} = \frac{28}{77}.$$

E and F of the above example are not independent because

$P(E \cap F) = \frac{28}{81}$ and $P(E) \cdot P(F) = \frac{77}{81} \cdot \frac{28}{81} \neq P(E \cap F)$.

Fourfold contingency table

A fourfold contingency table can be used for the representation and calculation of the probabilities for random experiments with two different characteristic attributesM_1 and M_2. In doing so, $\overline{M}_1$ ("not M_1") represents the complement toM_1 and $\overline{M}_2$ ("notM_2") represents the complement toM_2 .

	M_1	$\overline{M}_1$	sum
M_2	$P(M_1 \cap M_2)$	$P(\overline{M}_1 \cap M_2)$	$P(M_2)$
$\overline{M}_2$	$P(M_1 \cap \overline{M}_2)$	$P(\overline{M}_1 \cap \overline{M}_2)$	$P(\overline{M}_2)$
sum	$P(M_1)$	$P(\overline{M}_1)$	1

Example:

In the urn above, the 9 balls are numbered 1 through 9 and the orange balls receive the numbers 1 and 2. In addition to the color, it is also recorded whether the labeled number is even or odd.

Number of possibilities			
	even	odd	sum
green	3	4	7
orange	1	1	2
sum	4	5	9

Probabilities			
	even	odd	sum
green	$\frac{3}{9}$	$\frac{4}{9}$	$\frac{7}{9}$
orange	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$
sums	$\frac{4}{9}$	$\frac{5}{9}$	1

5.3. Statistical Parameters

If every outcome of a random experiment is assigned a number, one refers to a **random variable** X with values x_i. $P(X = x_i)$ designates the probability that the random variable X will assume value x_i.

One possible random variable in the above urn example is X: The number of orange balls drawn.

Outcome	gg	go	og	oo
Value x_i of random variable X	0	1	1	2

The probability distribution of the random variable can now be clearly depicted in a table.

Value x_i of random variable X	0	1	2
$P(X = x_i)$	$\frac{49}{81}$	$\frac{14}{81} + \frac{14}{81} = \frac{28}{81}$	$\frac{4}{81}$

The **expected value** μ is that value which the random variable assumes on average. It corresponds to the average value which is calculated for quantitative characteristics and is calculated analogously by substituting the relative frequencies with the probabilities.

$$\mu = x_1 \cdot P(X = x_1) + x_2 \cdot P(X = x_2) + \cdots + x_k \cdot P(X = x_k)$$

In the example the expected value is $\mu = 0 \cdot \frac{49}{81} + 1 \cdot \frac{28}{81} + 2 \cdot \frac{4}{81} = \frac{36}{81} = \frac{4}{9}$.

Just as with the average, an important parameter related to the expected value is the variance. Again, the squaring prevents positive and negative divergences from the expected value from cancelling each other out. And again, this means that the variance does not have the same units as the expected value.

$$\sigma^2 = (x_1 - \mu)^2 \cdot P(X = x_1) + (x_2 - \mu)^2 \cdot P(X = x_2) + \cdots + (x_k - \mu)^2 \cdot P(X = x_k)$$

The variance for the random variable above is therefore

$$\sigma^2 = \left(0 - \frac{4}{9}\right)^2 \cdot \frac{49}{81} + \left(1 - \frac{4}{9}\right)^2 \cdot \frac{28}{81} + \left(2 - \frac{4}{9}\right)^2 \cdot \frac{4}{81} \approx 0\ 38.$$

Analogous to the case with the average value, the **standard deviation** $\sigma = \sqrt{\sigma^2}$ can be used here as a measure of the scattering about the expected value. The standard deviation always has the same units as the expected value and in this case$\sigma \approx 0\ 62$ applies.

Part II.

Exercises

6. Fundamentals

6.1. Arithmetic Training

All of the exercises for this section (Arithmetic Training) are originally from : *Abele. Kammermeyer. Mohry und Zerpies;* ***Mathematik 5./6. Klasse****: Richtig Mathematik lernen. Komet Verlag.* 2007. You should be able to solve these exercises (without a calculator) in less than 60 minutes. Practice these elementary calculations until they come **easy** to you.

1. Write the following in exponential form and calculate the value of the exponential:

   a) $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$ c) $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ e) $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ g) $100 \cdot 100 \cdot 100$

   b) $5 \cdot 5 \cdot 5 \cdot 5$ d) $12 \cdot 12$ f) $25 \cdot 25 \cdot 25$

2. Calculate and reduce the following completely:

   a) $\frac{65}{91}$ b) $\frac{52}{78}$ c) $\frac{88}{99}$ d) $\frac{3}{10} + \frac{4}{7}$ e) $\frac{5}{18} + \frac{7}{30}$ f) $\frac{7}{10} + \frac{6}{45}$

3. Write the following as the sum of a whole number and a real fraction:

   a) $\frac{193}{9}$ b) $\frac{432}{25}$ c) $\frac{647}{120}$

4. Answer the following questions:

   a) Which number must you divide by $\frac{7}{8}$ in order to obtain $\frac{4}{5}$?

   b) By which number must you divide $\frac{3}{7}$ in order to obtain $\frac{9}{14}$?

5. Solve for x as a reduced fraction, a decimal number, and as a percentage:

   a) $4x = 9$ b) $15x = 10$ c) $48x = 18$

6. Calculate the following:

   a) $\left(\frac{3}{4}\right)^2 \cdot \left(\frac{4}{5}\right)^3$ b) $\left(4 + \frac{1}{2}\right)^2 \cdot \left(\frac{1}{3}\right)^3$

7. Perform the division on paper. write down any remainder as an additive fraction:

   a) $10775 : 25$ b) $12345 : 23$ c) $4601553 : 451$

8. Divide in your head (applying the same principle as in the preceding exercise):

   a) $\dfrac{1}{99}$ b) $\dfrac{15}{14}$

9. Solve the following equations for x:

   a) $354 \cdot 13 - x = 756 : 18$

   b) $421 \cdot x - (97 + 48) \cdot x = (15 \cdot (94 - 79) + 63) \cdot 23$

   c) $\frac{8}{25} \cdot x = 10\,\mathrm{dm}^2$ (x is a decimal number incm^2)

10. Answer the following questions:

    a) What is the average velocity of an ICE train which requires exactly 45 minutes for 153 km?

    b) Mr. Hurtig fills the 45 liter fuel tank of his automobile. The gas station attendant gives him € 33.50 change out of € 100. How much fuel was in the tank before Mr. Hurtig gassed up if a liter of fuel costs €1.75?

    c) What is the maximum number of intersections that three nonidentical nonparallel lines have in space? What is the minimum number of intersections?

11. Calculate the surface and the volume of the cuboid with edge lengths : 45 dm. 4 m. 125 cm.

12. Break the following down into prime factors:

    a) 425 b) 248 c) 336 d) 225

13. Arrange the fractions by size (Hint: Compare. do not calculate.):

    a) $\frac{4}{9}, \frac{4}{7}, \frac{4}{11}$ b) $\frac{7}{15}, \frac{14}{17}$ c) $\frac{14}{25}, \frac{31}{20}, \frac{4}{15}$

14. Write the following numbers as fully reduced fractions:

    a) 0 125 b) 1 375 c) 1 75 d) 2 625 e) 1 875

15. Calculate:

    a) $\dfrac{\left(15 \cdot \frac{3}{20} + \frac{48}{55}\left(3 + \frac{7}{16}\right)\right) : \left(3 + \frac{1}{2}\right)}{\left(1 + \frac{9}{10}\right) : \left(1 + \frac{13}{25}\right) + \frac{3}{4} : 2}$ b) $\dfrac{\left(2 + \frac{1}{3}\right)^2 - \left(1 + \frac{5}{6}\right)^2 + \frac{1}{6}}{\left(4 + \frac{2}{5}\right)\frac{5}{12} + 2 + \frac{2}{3}}$

16. Calculate:

    a) $2\ 74 \cdot 1\ 08$ b) $6\ 55 \cdot 1\ 58$ c) $40\ 11 : 7$ d) $30 : 6{,}4$ e) $1 : 7$

6.2. Mathematical Operations and Transformations

1. Write the expressions without parentheses and simplify as much as possible.

   a) $x^2(x^3 - 2x) - x^5$
   b) $x(x-1) - x^2$
   c) $(3x+2)^2$
   d) $3x(2-3x)$
   e) $x + 1 - 2(x-1)$
   f) $2(x^2-1) - x^2$
   g) $1 - (1-3x)^2$
   h) $2x^2(2x^2+1)$
   i) $2(x+1) - 3(x+2)$
   j) $(x - x^3)x^2$

2. Write the expressions without parentheses and simplify as much as possible:

a) $(9+4x-a)\cdot(-4)$

b) $(5a+4b)\cdot(6x-7y)$

c) $7x\cdot(2a-3b-4c)\cdot 2y$

d) $(4y+6x)\cdot(3a-5b)-(2y+3x)\cdot(2a+3b)$

e) $(15xy+12bx)\cdot(a-c)-(5bx-10x)\cdot(a-c)$

f) $2n\cdot(3x+z)-(9x+3z)\cdot(2n+3)-3x-z$

3. Skillfully combine the following :

a) $23u-(14v-(8v+6u-3v-(43v-16u))-16u)$

b) $(3p-2q)^2-(3p+2q)^2$

c) $(2a+b-c)x+(c-2a-b)y$

6.3. Calculating Fractions

1. Combine the fractions into a single fraction.

a) $\frac{9}{4}\left(-\frac{8}{9}-\left(\frac{1}{3}+\frac{1}{6}\right)\right)$

b) $\left(\frac{3}{2}\cdot\frac{2}{3}-\left(\frac{1}{2}+\frac{1}{7}\right)\right)\cdot\frac{14}{15}$

c) $\left(\frac{3}{5}-\frac{1}{25}\right):\left(1-\left(\frac{14}{50}-\frac{3}{100}\right)\right)$

d) $\frac{7}{2}+\left(\left(\frac{3}{2}+\frac{15}{4}\right):\frac{1}{16}\right)$

e) $\frac{2}{3}:\left(\frac{4}{9}:\frac{3}{2}+\left(\frac{1}{2}-\frac{2}{3}:\frac{9}{4}\right)\right)$

f) $\frac{49}{5}-\left(\frac{9}{10}:\frac{3}{8}\right):\left(\frac{1}{4}\cdot\left(\frac{12}{13}\cdot\frac{13}{12}\right)\right)$

2. Simplify as much as possible:

a) $\dfrac{1}{y}=\dfrac{1}{x^2+1}+C$ *(solve for y)*

b) $\dfrac{u-v}{v-u}$

c) $\dfrac{mt+ms-nt-ns}{mt-ms-nt+ns}$

d) $\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right):\dfrac{1}{abc}$

e) $\dfrac{1}{x+\dfrac{1}{2x-\dfrac{2x^2}{1+x}}}$

6.4. Exponent and Root Arithmetic, Logarithms

1. Simplify the expressions as much as possible.

a) $a^{m-n+1}\cdot a^{m+n-8}$

b) $x^{3n-6}\cdot 3x^{n-3m+2}\cdot 5x^{2-n}$

c) $\sqrt[4]{a^{n+4}}:\sqrt{a^{n-4}}$

d) $\sqrt{\sqrt[3]{x^8}}$

e) $\left(\dfrac{x^{-4}y^{-5}}{a^{-1}b^3}\right)^2\cdot\left(\dfrac{x^3a^{-2}}{y^2b^2}\right)^{-3}$

f) $\dfrac{45xa^3}{9yb^3}\cdot\dfrac{9y^n(a-1)^2}{30x^n(a+1)^2}:\dfrac{9y^{n-1}(1-a)^3}{24x^{n+1}(1+a)^2}$

2. Solve the equations for x.

a) $16x+19=5(4+3x)$

b) $\dfrac{5}{2x}-\dfrac{3}{6x}=\dfrac{1}{3}-\dfrac{3}{x}$

c) $5(2x+3)-12(6-x)=11(4x+7)$

d) $4x-15(x-1)=2(6-3x)$

e) $\dfrac{12}{15x}+\dfrac{2}{3}=\dfrac{3}{5}+\dfrac{2}{3x}$

f) $\dfrac{x+1}{3}-\dfrac{2x+5}{6}=\dfrac{3-4x}{2}$

3. Solve the equations for x.

a) $-5+10x^2+5x+(x+2)(1-11x)=-21x-3$

b) $-2x^2+3x-3(-3x+7)=-6+(3-x)(-7+2x)$

4. Solve the root equations for x.

a) $3\sqrt[3]{x-1}=\sqrt[3]{x+207}$

b) $\sqrt{x}=\sqrt{x+8}-2$

c) $\sqrt{x-1}=x-7$

d) $\sqrt{x+2\sqrt{x+7}}=6$

e) $\sqrt{2x+\sqrt{4x-3}}=3$

f) $3\sqrt{x+\sqrt{x-4}}=6$

g) $\sqrt{4x+\sqrt{x+3}}=\sqrt{5x+1}$

h) $\sqrt[4]{x+2}=\sqrt[8]{4x+8}$

5. Calculate:

a) $\left((-2)^{-2}\right)^3$

b) $\left((-2)^3\right)^{-2}$

c) $(-2^3)^2$

d) $(-2^3)^{-3}$

e) $(-x^3)(-x^2)(-x)^4$

f) $\dfrac{(-2x)^{-2m}}{-2x^{-2m+1}}$

g) $\dfrac{1}{x}-\dfrac{2}{x^2}+\dfrac{1}{x^3}$

6. Solve the equations for x.

a) $3^{x+1}=5$

b) $5^{2x-1}=12$

c) $5\cdot 3^{x+2}=72$

d) $15\cdot e^{2x}=66$

e) $1-e^{5x}=0{,}3$

f) $5^{3x}=7^{2x}$

g) $10^{7-x}=6^{2x+1}$

h) $3^x\cdot 4^{x+1}=5^{x+2}$

i) $\log_x 81=\frac{1}{4}$

j) $\log_7 x=-2$

k) $\log_2 x=-8$

l) $80=16\cdot 4^x$

m) $1000=0{,}5^x$

n) $2^x=\frac{1}{\sqrt{\sqrt{2}}}$

o) $4^x=3{,}2$

p) $2^{-x}=-2^x$

7. Simplify the following:

a) $5\,(a-b)^{2k-2}\,\frac{9}{5}(b-a)^{7-2k}\cdot\frac{2}{3}(b-a)^{2k-5}$ where $k\in\mathbb{Z}$

b) $\dfrac{a-b}{\sqrt{a}-\sqrt{b}}-\dfrac{a-b}{\sqrt{a}+\sqrt{b}}$

c) $\sqrt{\dfrac{a}{b}\sqrt{\dfrac{b}{a}\sqrt{\dfrac{a}{b}}}}$

8. Calculate or split up the following:

a) $\log_a \frac{1}{a}$

b) $\log_a a$

c) $\log_a a^n$

d) $\ln\left(\sqrt[4]{a^3}\right)$

e) $\ln\left(\dfrac{x^2-y^2}{x^2+y^2}\right)$

f) $\ln e^e$

g) $\ln\sqrt{e}^e$

h) $\ln e^{\sqrt{e}}$

i) $\ln\left(\dfrac{u-v}{e(v-u)}\right)^2$

j) $\log_2 x=3$

k) $\log_x 2197=3$

6.5. Numbers and Sets

1. Assign the following numbers to the appropriated fundamental sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$:

$\sqrt{7}$, 12, -8, π, $\frac{2}{3}$, 1,41

2. Given three sets: $M_1=\{a,b,c,d,e\}$. $M_2=\{e,f,g,h,i\}$ and $M_3=\{a,c,e,g,i\}$. Determine the sets $M_1\cap M_2$. $M_1\cup M_2$. $M_1\cap M_3$. $M_1\cup M_3$. $M_2\cap M_3$ and $M_2\cup M_3$. In addition, also determine $M_1\setminus M_2$. $M_2\setminus M_1$. $M_2\setminus M_3$ and $M_1\cap M_2\cap M_3$ as well as$M_1\cup M_2\cup M_3$.

3. Given the following subsets of the real numbers:

$$A = \{x \in \mathbb{R} \mid -5 < x < 2\}, B = \{x \in \mathbb{R} \mid 4 > x \geqslant 0\},$$
$$C = \left\{x \in \mathbb{R} \mid x^2 \leqslant 9\right\}, D = \left\{x \in \mathbb{R} \mid x^2 > 1\right\}$$

Determine the following sets. You can first sketch the sets on the number line if you are having difficulties.

a) $A \cup B$
b) $A \backslash B$
c) $A \cap B$
d) $C \cap D$
e) $(\mathbb{R} \backslash D) \cup B$
f) $C \backslash (A \cap B)$
g) $(A \cap B) \cup (C \cap D)$

4. Solve the following for x and determine the solution set.

a) $4x + 17 > -x - 3$
b) $-4 \cdot (3x - 2) < 6 \cdot (1 - 2x)$

6.6. Quantities and Units

1. Which of the following units are coherent or incoherent, respectively?

a) Newton
b) Kilonewton
c) Millinewton
d) Kilogram
e) Kilometer
f) Nanosecond

2. Convert all given distances into kilometers.

a) 12 m
b) 3.4 m
c) 0.5 m
d) 100 mm
e) 120 dm
f) 150 mm
g) 5000 m
h) 200000 μm

3. Convert all given distances into millimeters.

a) 12 km
b) 120 m
c) 0.5 m
d) 100 cm
e) 50 km
f) 500 m
g) 800 m
h) 400 cm

4. Maritime travel uses the sea mile as the unit of length and the knot as the unit for velocity, where1 sea mile = 1sm = 1852m and 1 knot = 1kn= 1 sm/h. Convert the velocity 72 kn into the units km/h and m/s.

5. How many minutes and seconds are missing from 7 min 12 sec in order to round out a complete hour?

6. An indefatigable tortoise completes a trek of 100.8 km in a time of two weeks. Calculate the tortoise's velocity in km/h and in cm/s.

7. On a road map with scale 1: 250000 the distance to be driven is 12.3 cm long. How much time is will be required to drive this distance at an average velocity of 50 km/h?

8. A model railroad has a scale of 1: 125.

   a) Which distance must the little train travel in an hour if the actual velocity should correspond to 80 km/h?

b) Which distance does the little train travel in one minute and which in five minutes?

c) Which distance would the actual train have travelled if the little train has travelled 300 cm in 10 s?

d) Which velocity in km/h would the actual train then have reached?

9. Area calculation

   a) The old Federal Republic of Germany had an area of 248,800 km^2 and a population of 61.6 million. Calculate the population density (number of inhabitants per km^2)

   b) Canada has an area of 9,976,000 km^2 and a population of 24 million. What is the ratio of the population density of the old Federal Republic of Germany to that of Canada's?

   c) How often does the area of the old Federal Republic of Germany fit into the area of Canada?

10. The speed of sound is approximately 333 m/s. What is the speed of an airplane in

    a) km/h and m/min at twice the speed of sound?

    b) km/h and m/min at four times the speed of sound?

6.7. Cross-Multiplication

1. 12 pieces of work produce 3.3 kg of shavings. How many kg of shavings result from 100 pieces of work?

2. A container holds 450 l of water and is filled from an open tap in 12 minutes. How many liters of water were in the container after 7 minutes?

3. A commercial airliner with a velocity of 850 km/h travels a certain distance in 55 minutes. How much time does a new type of airplane which can travel at 950 km/h require for the same distance?

4. Find the error in the calculation. The final price for an automobile is €23,800. What is the price before taxes?

$$\begin{array}{rr} 100 & 23800 \\ :100 \downarrow & :100 \downarrow \\ 1 & 238 \\ \cdot 19 \downarrow & \cdot 19 \downarrow \\ 19 & 4522 \end{array}$$

   The price before taxes is € 23,800 - € 4522 = € 19,278.

6.8. Systems of Linear Equations

Determine the solutions to the following systems of linear equations.

1. $5x - 2y = 24$
   $x + 3y = -2$

2. $2x - 2y = 2$
   $-x + y = -2$

3. $y = 4x - 7$
   $-8x + 2y = -14$

4. $b + 3c = 10$
   $5a - 3b + c = 5$
   $-2b + 2c = -4$

5. $x + 2y + z = 9$
   $-2x - y + 5z = 5$
   $x - y + 3z = 4$

6. $4x + y + 7z = 12$
   $5x + 10z = 5$
   $-x - 2y = -2$

7. $x - y + z = -2$
   $4x + 2y + z = -5$
   $6x + 3z = -9$

6.9. For Experienced Students

1. Do the arithmetic in your head: Begin with the given number and follow the arithmetic directions from top to bottom. Try to be skillful in your calculations. The degree of difficulty increases from left to right. If you are experienced, you should require less than a minute per column.

Simple	**Intermediate**	**Difficult**
40	57	98
take $\frac{1}{4}$	$\cdot 3$	$+524$
square it	$+211$	$+\frac{1}{2}$ of the value
$+24$	take half	$/3$
$/4$	-52	-243
$\cdot 2$	double it	$\cdot 5$
-30	$+122$	$/20$
$/4$	$/20$	square it
$\cdot 7$	$+\frac{3}{10}$ of the value	$+473$
-22	$/13$	$+\frac{2}{3}$ of the value

2. Determine the solution set of the following equations.

a) $\dfrac{9}{x-8} = x$

b) $\dfrac{5}{x-2} - 3 = \dfrac{2x-4}{5}$

c) $\dfrac{2x+1}{3} + \dfrac{10}{2x+1} = 4$

d) $\dfrac{2x}{x-4} + \dfrac{3x}{x+4} = \dfrac{4(x^2-x+4)}{x^2-16}$

3. Determine the solution set dependent on p.
   $5 \cdot (3 + x) > p \cdot (1 - x) + 3 \cdot (x + 5)$

4. A farmer wants to dry freshly harvested fruit. He spreads approximately 100 kilograms of it out on a large tarpaulin and lets the sun go to work. Initially, the water content is at 99 percent.
   A few days later the water content has been reduced to 98 percent. How much does the fruit weigh now - including all the water it still contains?

5. Determine the solutions to the following systems of equations.

a) $-4x + 7y - 5z = 4$
   $6x + 3z = -8$
   $-4x - 8y - 6z = -8$

b) $2a + 6b - 3c = -6$
   $4a + 3b + 3c = 6$
   $4a - 3b + 9c = 18$

c) $2x - 3y - z = 4$
   $x + 2y + 3z = 1$
   $3x - 8y - 5z = 5$

7. Functions

7.1. Linear Functions

1. Determine the equation of the form $f(x) = ax + b$ based on the given information.

   a) $a = 3$ and the point $P(2; 16)$ lies on the line.

   b) The line goes through the point $Q(-1; -2)$ and is parallel to the line with equation $g(x) = -2x + 1$.

   c) It is $b = -1{,}5$ and $f(2{,}5) = 16$.

   d) It is $f(3) = 4$ and $f(-9) = 8$.

2. A company manufactures chairs. The production costs consist of fixed costs (rent, machines, ...) amounting to 20,000 Euros and variable costs (materials, labor, ...) amounting to 30 Euros for each individual chair. Chairs are sold for 80 Euros apiece.

   a) Determine the function describing the profit (income minus costs) as a function of the number of chairs.

   b) How high is the profit if 3000 chairs are sold?

   c) As of what number of sold chairs does the company begin to make a profit?

7.2. Quadratic Functions

1. Solve for the roots and complete the square in order to determine the vertex form for function $f(x)$.

   a) $f(x) = x^2 - 4x + 1$ b) $f(x) = x^2 + 3x - 4$ c) $f(x) = -2x^2 + 4x - 8$

2. The quadratic function $f(x) = ax^2 + bx + c$ has the roots -1 and 3. For the following, determine the parameters a, b and c for the following valued of f at point $x = 0$:

   a) $f(0) = 3$ b) $f(0) = -3$ c) $f(0) = -6$ d) $f(0) = 1\ 5$

7.3. Rational Functions (Polynomials)

1. Solve the following equations for x. Make certain that it is clear to you for which function you have determined the roots by doing so.

   a) $x^2 + x = 0$

   b) $(x-3)^2 = 16$

   c) $\dfrac{1}{x^2} - \dfrac{1}{x} - 2 = 0$

   d) $\dfrac{1-x}{1+x} = x$

   e) $a - (a-b)\,x = (b-a)\,x - (c+bx)$

   f) $\sqrt{2x^2 - 1} = -x$

2. Determine the roots of the functions.

   a) $f(x) = x^3 - 2x^2 - 11x + 12$

   b) $f(x) = 2x^3 - 6x^2 + 8$

   c) $f(x) = -\frac{1}{2}x^3 - 6x^2 - 4{,}5x + 11$

   d) $f(x) = x^4 - 4x^3 - 13x^2 + 4x + 12$

   e) $f(x) = x^3 - 3x^2 - 2x - 40$

   f) $f(x) = 20x^3 - 120x^2 + 220x - 120$

7.4. Trigonometric Functions

1. Determine the angle in radians as a multiple of π .

   a) 180°, 270°, 315°, 135°

   b) 2°, 10°, 100°, 60°

2. Determine the angle in degrees.

   a) 3π, $\frac{3}{10}\pi$, $\frac{\pi}{2}$, $\frac{2}{3}\pi$

   b) $\frac{\pi}{6}$, $\frac{\pi}{18}$, $\frac{5}{18}\pi$, $\frac{5}{4}\pi$

3. Solve for the exact value of the function.

   a) $\sin\left(\frac{3}{2}\pi\right)$

   b) $\sin\left(-\frac{7}{4}\pi\right)$

   c) $\cos\left(\frac{2}{3}\pi\right)$

   d) $\cos(-3\pi)$

4. Solve for the period and amplitude of f. In addition, without calculating, determine the coordinates of one maximum H and one minimum T of the graph of f.

   a) $f(x) = 2\sin(2x)$

   b) $f(x) = \sin(\pi(x-1)) + 1$

   c) $f(x) = \cos\left(\frac{\pi}{2}x\right) - 1$

   d) $f(x) = -1\ 5\cos(x+\pi)$

5. Solve.

   a) $\sin(2x) + 7 = 8,\ x \in [0; 2\pi]$

   b) $\cos\left(\frac{\pi}{4}x\right) - 1 = 0,\ x \in [0; 10]$

7.5. The Inverse Function

1. Determine the domain $D_f \subseteq \mathbb{R}$ of the function $f : D_f \to \mathbb{R}$ defined by $f(x)$ so that it can be unambiguously inverted. Determine the inverse function and its domain.

   a) $f(x) = \dfrac{1}{x-1}$

   b) $f(x) = \dfrac{x+3}{5x-7}$

   c) $f(x) = \sqrt{2x+6}$

2. Determine the inverse function of each of the following functions $f : (0, \infty) \to \mathbb{R}$, each in turn defined by $f(x)$. Note: $x > 0$ applies in each case.

   a) $f(x) = 3x^2 + x$
   Note: Vertex form!

   b) $f(x) = (x^2+1)^{-1}$

   c) $f(x) = \sqrt{-2x}$

7.6. Root Functions

Solve for the zeroes of the following functions.

1. $f(x) = x^4 - 625$
2. $f(x) = 2x^3 + 0{,}25$
3. $f(x) = \frac{1}{8}x^4 + 2$
4. $f(x) = (x-3)^5 + \frac{1}{32}$
5. $f(x) = \sqrt[3]{x} + 8$
6. $f(x) = \sqrt[5]{x-1} - 2$
7. $f(x) = x^{-1} - 5\sqrt{x^3}$

7.7. Exponential Functions and Logarithms

1. Solve exactly for the zeros of the following functions.

   a) $f(x) = e^x - 5$

   b) $f(x) = -3e^x + 2$

   c) $f(x) = e^{-x} + 4$

   d) $f(x) = \frac{5}{2}e^{-2x} - \frac{5}{2}e$

2. Determine the exponential function of form $f(x) = a \cdot q^x$ for the function whose plot goes through the points $P = (0; 1\ 5)$ and $Q = (2; 6)$.

3. A nutrient solution initially contains 50,000 bacteria at the beginning of an experiment. The number of bacteria increases by 10% daily.

   a) Determine the corresponding growth function.

   b) How many bacteria are in the nutrient solution after five days?

   c) Determine the length of time it takes for the number of bacteria to double.

   d) After how much time has the number of bacteria increased by a factor of ten?

7.8. Symmetry Relations of Functions

Determine which symmetry relations the following functions $f : D_f \to \mathbb{R}$ have:

1. $f(x) = x^3 - 5x$
2. $f(x) = \sqrt{x}$
3. $f(x) = \sin(x)$
4. $f(x) = |x|$
5. $f(x) = \sin(x + \pi)$

7.9. Translation, Stretching and Compression

1. Perform the following translations and modifications on the unit parabola:

   a) Translation by 1 to the right

   b) Reflection about the x-axis

   c) Stretching by a factor 2 in the $y-$direction

   d) Translation upwards by 1

   Sketch the graph of the function. Calculate the zeros in order to do this. Write the equation for the function in the conventional form.

2. Translate the function $f(x) = 3e^{-5x} - 5$ so that the graph intercepts the x-axis at $x = 0$.

3. Stretch the function $f(x) = -e^{3x-3} - 2$ so that it has a value 4 at $x = 1$.

4. Determine the expression for the resulting function $g(x)$ when

   a) The amplitude of function $f(x) = \sin(x)$ is doubled and the graph of the function is then translated downwards by 3.

b) The amplitude of function $f(x) = \cos(x)$ is reduced to a third and the period is quadrupled.

c) The graph of the function $f(x) = \sin(x)$ is translated upwards by 1 and to the left by 4.

7.10. Combination and Composition of Functions

1. Determine $(f \circ g)(x)$ and $(g \circ f)(x)$ and the corresponding domains!

   a) $f(x) = 1 + x^2, \quad D_f = \mathbb{R}$ and $g(x) = \sqrt{x}, \quad D_g = [0; \infty)$

   b) $f(x) = (3 - x)^2, \quad D_f = \mathbb{R}$ and $g(x) = 3x + 1, \quad D_g = \mathbb{R}$

   c) $f(x) = -e^x, \quad D_f = \mathbb{R}$ and $g(x) = x^2 - 3, \quad D_g = \mathbb{R}$

2. Determine a possible composition of form $h \circ g$ for function f.

   a) $f(x) = \sin(x - 3)$

   b) $f(x) = (2x + 8)^3$

   c) $f(x) = \dfrac{1}{x^2 - 7}$

   d) $f(x) = e^{2x}$

7.11. For Experienced Students

1. A chess club is hosting a large chess tournament. Each participant plays exactly one match against every other participant. After each match, the players are given cards. The winners are given green cards, the losers are given red cards. In case of a tie, each player receives a yellow card.
   After the tournament, the organizer determines that exactly 752 cards of each color were distributed. How many participants took part in the tournament?

2. Using the two building blocks $x + 3$ and $x^2 - 4$, construct a rational function

   a) of degree 3 with 3 simple zeros,

   b) of degree 4 with 2 double zeros,

   c) of degree 5 with exactly one zero.

3. Solve

   a) $-2\cos(\pi(x - 1)) + 1 = 2, \; x \in [-2; 2]$

   b) $-2\cos(x + \pi) = \sqrt{3}, \; x \in [0; 2\pi]$

4. Determine the exponential function of form $f(x) = a \cdot q^x$ whose graph goes through the points P(-2/50) and Q(3/0,512).

5. The amount of active ingredient of a pain killer declines approximately exponentially inside a human body. If a patient takes a tablet containing 0.5g of the active ingredient, approximately 0.09g remain after 10 hours.

   a) After what period of time has the amount of active ingredient declined by half (by 90%) ?

   b) A person takes one tablet at 9am and a second tablet at 3pm, each containing 0.5g of the active ingredient. How many grams of the active ingredient are still present in the body at 8pm on the same day?

8. Differential and Integral Calculus

8.1. Differential Calculus

1. Determine the first and second derivatives of the functions.

   a) $f(x) = 5x^5 - 3x^4 - 2x^2 - 3$

   b) $f(x) = 2(3x^3 - 2x)$

   c) $f(x) = x^3 \cdot (2x^2 - 4)$

   d) $f(x) = 2x^3 \cdot 4x^2$

   e) $f(x) = \dfrac{2x^2 + 3x}{4x^3}$

   f) $f(x) = \dfrac{x^4 - 1}{2x}$

   g) $f(x) = \dfrac{1}{x + 1}$

   h) $f(x) = (x + 1)^7$

   i) $f(x) = (x^2 - 1)^{-3}$

2. Derive the following functions.

   a) $f(x) = \sqrt{3x - 4}$

   b) $f(x) = (\sin x)^2$

   c) $f(x) = \cos x^2$

   d) $f(x) = \sqrt[3]{2x^2 + 3x}$

   e) $f(x) = x^3 \cdot \sin x$

   f) $f(x) = \dfrac{3x^2}{\cos x}$

8.2. Applications for Differential Calculus

1. Determine the first derivative of the following functions and find the equation for the tangent at point $x_1 = 2$

   a) $f(x) = 2x^4$

   b) $f(x) = \dfrac{1}{x^2}$

   c) $f(x) = x^{-3}$

   d) $f(x) = \left(\frac{1}{4}x\right)^3$

2. Determine the extreme points for the following functions and decide for each respective case whether it is a local maximum or a local minimum.

   a) $f(x) = \frac{1}{3}x^3 + \frac{7}{4}x^2 - 2x - 3$

   b) $f(x) = e^x + e^{-x}$

   c) $f(x) = 2\sin(\pi x),\ x \in [0; 2]$

8.3. Integral Calculus

1. Calculate

$$\int f(x)\,dx$$

for

a) $f(x) = 2x^4$

b) $f(x) = \dfrac{1}{x^2}$

c) $f(x) = \left(\frac{1}{4}x\right)^3$

d) $f(x) = 2x^4 + 4x^3 - 3x^2$

e) $f(x) = 5x^5 - 3x^4 - 2x^2 - 3$

f) $f(x) = x^3 \cdot (2x^2 - 4)$

g) $f(x) = 2x^3 \cdot 4x^2$

h) $f(x) = (2x^2 + 3x)\left(4x^3\right)^{-1}$

i) $f(x) = \dfrac{x^4 - 1}{2x}$

j) $f(x) = \dfrac{1}{x+1}$

k) $f(x) = (x+1)^7$

l) $f(x) = \sqrt{x-4}$

m) $f(x) = \sin x$

n) $f(x) = \dfrac{3}{\sqrt[5]{x}}$

o) $f(x) = \sqrt{x}\left(\sqrt{x} - 1\right)^2$

2. Calculate

a) $\int_0^1 (x - x^2)\,dx$

b) $\int_1^4 2(3x^3 - 2x)dx$

c) $\int_{\frac{1}{5}}^5 x^{-4}dx$

3. Calculate the area underneath the curve of $f : \mathbb{R} \to \mathbb{R}$ for $x \geqslant 0$, $f(x) \geqslant 0$ where

a) $f(x) = 4 - x^2$

b) $f(x) = e - e^x$

8.4. For Experienced Students

1. Determine a polynomial function of the lowest degree possible for which the following applies:

   a) The function has a maximum at H=(0;1).

   b) The graph of the function intercepts the x-axis at $x = 2$.

   c) There is a point of inflection at $x = 1$.

2. Which function $f(x) = a \cdot \sin(\pi \cdot x) + b$ has slope 3π and value 4 at $x = 2$?

3. Determine a, b and k for $f(x) = a \cdot e^{kx} + b$ under the following conditions:

   a) The asymptote of the graph of the function is described by the equation $y = 3$.

   b) The graph of the function intercepts the y-axis at -2 .

   c) The tangent at point $x = 0$ has slope $5 \cdot \ln(2)$.

Vector Algebra

8.5. Geometry Equations

Calculate the volume as well as the surface area

1. of the cuboid with edge lengths: L= 100 m; B= 100 cm; H= 50 mm.
2. the cylinder with diameter D= 5 km and height H= 500 mm.

8.6. Mathematical Operations with Vectors

1. A straight prism ABCDEF, has A, B, and C as the corners of the base. The height of the prism is 5. Determine the coordinates of points D, E, and F if
   a) A=(2;0;3), B=(1;0;7), C=(-7;0;3)
   b) A=(2;0;3), B=(6;2;3), C=(3;3;3)
   c) Which special position do points A, B, and C have in the coordinate system?
2. Calculate with the following vectors and scalars:
$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \vec{v} = \begin{pmatrix} \frac{3}{2} \\ -3 \\ 0 \end{pmatrix}, \vec{w} = \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix}, k = 5, l = -2$$
   a) $\vec{u} + \vec{v} + \vec{w},\ \vec{u} - \vec{v} - \vec{w},\ \vec{u} - \vec{v} + \vec{w}$
   b) $l\vec{w},\ k\vec{u} + l\vec{v},\ k\vec{v} - l\vec{v}$
   c) a, b, so that $a\vec{u} + b\vec{v} = \vec{w}$ holds. What does the result tell you?
   d) The length of all three vectors.
3. Use vectors to determine the midpoint of segment AB.
   a) A=(3;2;5), B=(5;2;3) b) A=(2;1;-2), B=(-5;1;9) c) A=(0;0;2), B=(-2;0;0)
4. For triangle ABC, pointsM_A, M_B and M_C are the midpoints of the triangle edges which lie directly opposite the respective corners. Determine the coordinates of points M_A, M_B and M_C as well as the sum of the vectors $\overrightarrow{AM_A}$, $\overrightarrow{BM_B}$ and $\overrightarrow{CM_C}$.
   a) A=(0;0), B=(3;1), C=(1;3) b) A=(0;0;0), B=(3;1;2), C=(1;3;4)
5. Points P=(1;3;5) and Q=(2;-1;7) are given.
   a) Determine at least 2 different equations for the straight line through P and Q.
   b) Determine whether point R=(2;-5;9) lies on the straight line.
6. Points A=(7;5;4), B=(-5;8;7) and C=(-1;1;3) are given.
   a) Show that triangle ABC is an isosceles right triangle.
   b) Calculate the area of the triangle.

7. Determine the relative position of line g to line h.

$$g : \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + r \cdot \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}, \ r \in \mathbb{R}, \ h : \vec{x} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix}, \ s \in \mathbb{R}$$

8. Determine the intersection of lines g and h.

$$g : \vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + r \cdot \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}, \ r \in \mathbb{R}, \ h : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + s \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \ s \in \mathbb{R}$$

9. Given the vectors $\vec{a} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$.

   a) Calculate the cross products $\vec{a} \times \vec{b}$, $\vec{a} \times \vec{c}$ and $\vec{b} \times \vec{c}$.

   b) Calculate the respective areas of the parallelograms whose adjacent sides are $\vec{a}$ and $\vec{b}$, $\vec{a}$ and $\vec{c}$, as well as $\vec{b}$ and $\vec{c}$, respectively.

   c) Determine the angle enclosed by $\vec{a}$ and $\vec{b}$. Use the scalar product to do so. Subsequently verify your result by using the vector product. Analogously, calculated the angle between $\vec{a}$ and $\vec{c}$, as well as between $\vec{b}$ and $\vec{c}$.

8.7. For Experienced Students

The straight line flight paths of two airplanes F_1 and F_2 can be specified using a coordinate system. Flight path F_1 is given by the points P=(2;3;1) and Q=(0;0;1,05) and flight path F_2 is given by R=(-2;3;0,05) and T=(2;-3;0,07). The coordinates specify the distances from the coordinate origin in kilometers. There is no wind. F_1 flies with a velocity of 350 km/h and F_2 with a velocity of 250 km/h relative to the air. The location of F_1 is at point P and at the same time the location of F_2 is at point R. Let us consider the situation 20 minutes later.

1. What is the location of the two airplanes? What is their altitude?
2. How far are the two airplanes apart?

9. Stochastic Processes

9.1. Relative and Absolute Frequencies

1. After an inspection of 2325 cattle, the relative frequency of animals infected with brucellosis was determined to be 0.04.

   a) How many of the inspected animals were infected with the bacteria?

   b) How exact is the value you have calculated?

2. In a random sample of 57 high school graduates, the attribute "final grade point average" was compiled. The following table contains the initial results.

1.7	2.7	3.5	2.0	2.7	2.4	3.1	3.3	1.8	3.3	3.2	3.6	1.0	3.0	1.8
3.3	2.9	3.2	3.0	2.5	2.7	1.8	2.4	3.6	3.3	3.3	2.7	2.7	3.0	1.9
3.2	2.6	2.2	1.8	3.1	3.2	3.3	3.0	3.2	3.1	2.2	2.3	3.1	1.9	2.7
1.6	2.2	2.3	3.3	2.2	3.1	1.8	3.2	2.2	1.5	3.0	2.6			

   a) Determine the relative frequencies associated with the attribute classes very good $(1\ 0 \leqslant x \leqslant 1\ 5)$, good $(1\ 6 \leqslant x \leqslant 2\ 5)$, satisfactory $(2\ 6 \leqslant x \leqslant 3\ 5)$ and sufficient $(3\ 6 \leqslant x \leqslant 4\ 0)$. Create a tally chart in order to do this.

   b) Display the distribution of the absolute frequencies in a bar chart and the distribution of the relative frequencies in a pie chart.

3. One third of the employees of a certain company earns € 1600 per month, one fifth earns € 2000 per month, one sixth earns € 2300 per month and the rest earns € 3000 per month.

   a) Determine the average monthly income.

   b) What is the standard deviation?

4. In order to obtain an indication of the average annual mileage of passenger vehicles, 50 auto owners were randomly selected and asked about the total distance (in 1000s of km) they drove in the past year. following table contains the initial results.

15	12	16	25	5	30	8	10	15	20	7	10	20	30	15	25	21
15	18	30	45	5	2	16	24	25	10	14	20	15	12	8	14	5
25	10	15	28	12	10	20	32	42	13	18	25	12	15	20	2	

   Determine the arithmetic average, the variance, and the standard deviation.

9.2. Arithmetic with Probabilities

1. According to a statistic presented by the German rail company, approximately 95 percent of long-distance trains run "on time", that is with a maximum of 5 minutes delay. Tim travels five times by long-distance train.

   a) He calculates the probability that at least one train is not on time using the formula $1 - 0{,}95^5$. Under what circumstances can he apply that formula?

   b) Why is the assumption which Tim makes not necessarily correct?

2. The letters M, U, and T are to be used to spell three-letter words in which each letter appears only once. The words do not have to make sense. We consider the event E: "There is a consonant at the beginning" and F: "The last letter is a T".

    a) Describe the events E and F as sets and determine their probabilities.

    b) Describe the events $E \cap F$ and $E \cup F$ with words and determine their probabilities.

    c) Analyze whether E and F are independent events.

3. A coin is tossed until one side appears for the second time. Determine the probability distribution for the random variable X: "Number of tosses" as well as the expected value and the standard deviation of X.

4. A student states: "If I roll a die twice, the probability is $\frac{1}{3}$ that the outcomes will include a six because for one roll of a die the probability is $\frac{1}{6}$." Is the student correct?

5. How high is the probability of rolling at least one six if six dice are being rolled? Where do you apply the independence of events in this calculation?

6. A country performs a statistical survey. The following results are determined for the population: The percentage of employed inhabitants is 60%. The percentage of politically interested inhabitants is 56%. The percentage of politically interested inhabitants who are not employed is 14%.

    a) Calculate the missing percentages in the table.

Grouping	employed	not employed	sum
politically interested		0.14	
not politically interested			
sum	0.60		1

    b) One person is selected at random from all the politically interested inhabitants. What is the probability that that person is employed?

    c) One person is selected at random from all the employed inhabitants. What is the probability that that person is politically interested?

9.3. For Experienced Students

1. There are two possible types of errors which can occur during the production of USB sticks: A defective memory chip (Ch) or a defective connector (Co). The probability of a connector being defective is 5%. The probability of a memory chip being defective is 20%.

    a) Create a fourfold contingency table representing the situation.

    b) What is the probability of producing a functional USB stick which has neither a defective memory chip nor a defective connector?

2. A die has two pips on four of its sides and five pips each on the remaining two sides. This die is rolled repeatedly until the sum of the rolls is at least eight. The random variable X describes the number of rolls required for this to occur. Use a tree diagram to calculate the values of the probability distribution of X and the average number of required rolls.

Part III.

Solutions

10. Fundamentals

10.1. Arithmetic Training

1. Write the following in exponential form and calculate the value of the exponential: **Solution:**

   a) $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^5 = 100\,000$

   b) $5 \cdot 5 \cdot 5 \cdot 5 = 5^4 = 625$

   c) $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1^5 = 1$

   d) $12 \cdot 12 = 12^2 = 144$

   e) $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64$

   f) $25 \cdot 25 \cdot 25 = 25^3 = 15\,625$

   g) $100 \cdot 100 \cdot 100 = 100^3 = 1\,000\,000$

2. Calculate and reduce the following completely: **Solution:**

   a) $\frac{65}{91} = \frac{5}{7}$

   b) $\frac{52}{78} = \frac{2}{3}$

   c) $\frac{88}{99} = \frac{8}{9}$

   d) $\frac{3}{10} + \frac{4}{7} = \frac{21}{70} + \frac{40}{70} = \frac{61}{70}$

   e) $\frac{5}{18} + \frac{7}{30} = \frac{25}{90} + \frac{21}{90} = \frac{46}{90} = \frac{23}{45}$

   f) $\frac{7}{10} + \frac{6}{45} = \frac{7}{10} + \frac{2}{15} = \frac{21}{30} + \frac{4}{30} = \frac{25}{30} = \frac{5}{6}$

3. Write the following as the sum of a whole number and a real fraction: **Solution:**

   a) $\frac{193}{9} = \frac{189}{9} + \frac{4}{9} = 21 + \frac{4}{9}$ Caution: Never write $21\frac{4}{9}$ for $21 + \frac{4}{9}$. Under certain circumstances, the prior cannot be distinguished from: $21 \cdot \frac{4}{9} = \frac{84}{9}$

   b) $\frac{432}{25} = 17 + \frac{7}{25}$

   c) $\frac{647}{120} = 5 + \frac{47}{120}$ How can you tell that $\frac{47}{120}$ cannot be reduced any further?

4. Answer the following questions:

   a) Which number must you divide by $\frac{7}{8}$ in order to obtain $\frac{4}{5}$?
   Solution: $x : \frac{7}{8} = \frac{4}{5} \Rightarrow x = \frac{4}{5} \cdot \frac{7}{8} = \frac{28}{40} = \frac{7}{10}$

   b) By which number must you divide $\frac{3}{7}$ in order to obtain $\frac{9}{14}$?
   Solution: $\frac{3}{7} : x = \frac{9}{14} \Rightarrow x = \frac{3}{7} \cdot \frac{14}{9} = \frac{2}{3}$

5. Solve for x as a reduced fraction, a decimal number, and as a percentage:

   a) $4x = 9$ **Solution:** $x = \frac{9}{4} = \frac{9}{4} \cdot \frac{25}{25} = \frac{225}{100} = 2{,}25 = 225\%$

   b) $15x = 10$ **Solution:** $x = \frac{10}{15} = \frac{2}{3} = 0,\bar{6} = 66,\bar{6}\%$

   c) $48x = 18$ **Solution:** $x = \frac{18}{48} = \frac{3}{8} = 0{,}375 = 37{,}5\%$

6. Calculate the following: **Solution:**

   a) $\left(\frac{3}{4}\right)^2 \cdot \left(\frac{4}{5}\right)^3 = \frac{3^2 \cdot 4^3}{4^2 \cdot 5^3} = \frac{3^2 \cdot 4}{5^3} = \frac{36}{125}$

   b) $\left(4 + \frac{1}{2}\right)^2 \cdot \left(\frac{1}{3}\right)^3 = \left(\frac{9}{2}\right)^2 \cdot \left(\frac{1}{3}\right)^3 = \frac{3^4 \cdot 1^3}{2^2 \cdot 3^3} = \frac{3}{4}$

7. Perform the division on paper. write down any remainder as an additive fraction:

   a) $10775 : 25$ **Solution:**

   $$\begin{array}{cccccccl} & 1 & 0 & 7 & 7 & 5 & :25 & = 431 \\ - & 1 & 0 & 0 & & & & \\ & & & 7 & 7 & & & \\ & & - & 7 & 5 & & & \\ & & & & 2 & 5 & & \\ & & & - & 2 & 5 & & \\ & & & & & 0 & & \end{array}$$

   b) $12345 : 23$ **Solution:**

   $$\begin{array}{cccccccl} & 1 & 2 & 3 & 4 & 5 & :23 & = 536 + \frac{17}{23} \\ - & 1 & 1 & 5 & & & & \\ & & & 8 & 4 & & & \\ & & - & 6 & 9 & & & \\ & & & 1 & 5 & 5 & & \\ & & - & 1 & 3 & 8 & & \\ & & & & 1 & 7 & & \end{array}$$

   c) $4601553 : 451$ **Solution:** 10203

8. Divide in your head (applying the same principle as in the preceding exercise): **Solution:**

   a) $\dfrac{1}{99} = 0,\overline{01}$

   b) $\dfrac{15}{14} = 1{,}0\overline{714285}$

9. Solve the following equations for x:

   a) $354 \cdot 13 - x = 756 : 18$ **Solution:** $x = 4560$

   b) $421 \cdot x - (97 + 48) \cdot x = (15 \cdot (94 - 79) + 63) \cdot 23$
   Solution: Once the equation form $276x = 288 \cdot 23$ has been obtained, it should be easy to recognize that dividing by 23 is helpful. Subsequently, divide by 12 and obtain : $x = 24$

   c) $\frac{8}{25} \cdot x = 10\,dm^2$ (x is a decimal number incm^2) **Solution:** $x = 3125\,cm^2$

10. Answer the following questions:

    a) What is the average velocity of an ICE train which requires exactly 45 minutes for 153 km? **Solution:** $204\,\frac{km}{h}$

    b) Mr. Hurtig fills the 45 liter fuel tank of his automobile. The gas station attendant gives him € 33.50 change out of € 100. How much fuel was in the tank before Mr. Hurtig gassed up if a liter of fuel costs €1.75? **Solution:** $7\,\ell$

    c) What is the maximum number of intersections that three nonidentical. nonparallel lines have in space? What is the minimum number of intersections? **Solution:** Max.: 3 Min.: 0

11. Calculate the surface and the volume of the cuboid with edge lengths: $45\,dm$, $4\,m$, $125\,cm$.
    Solution:
    Volume: $V = 450 \cdot 400 \cdot 125\,cm^3 = 450 \cdot 50\,000\,cm^3 = 22\,500\,000\,cm^3 = 22{,}5\,m^3$
    Surface: $O = 2 \cdot (4{,}5 \cdot 4 + 4 \cdot 1{,}25 + 4{,}5 \cdot 1{,}25)\ m^2 = 2 \cdot 28{,}625\,m^2 = 57{,}25\,m^2$

12. Break the following down into prime factors: **Solution:**

    a) $425 = 5 \cdot 5 \cdot 17$ b) $248 = 2 \cdot 2 \cdot 2 \cdot 31$ c) $336 = 2^4 \cdot 3 \cdot 7$ d) $225 = 3^2 \cdot 5^2$

13. Arrange the fractions by size (Hint: Compare. do not calculate.):

    a) $\dfrac{4}{9}, \dfrac{4}{7}, \dfrac{4}{11}$ **Solution:** $\dfrac{4}{11} < \dfrac{4}{9} < \dfrac{4}{7}$ Hint: Same numerator.

b) $\frac{7}{15}, \frac{14}{17}$ **Solution:** $\frac{7}{15} < \frac{14}{17}$

c) $\frac{14}{25}, \frac{31}{20}, \frac{4}{15}$ **Solution:** $\frac{4}{15} < \frac{14}{25} < \frac{31}{20}$ Hint: *Significantly less than 0.5* is less than *somewhat more than 0.5* and that is less than *somewhat more than 1.5*

14. Write the following numbers as fully reduced fractions: **Solution:**

a) $0{,}125 = \frac{1}{8}$ b) $1{,}375 = \frac{11}{8}$ c) $1{,}75 = \frac{7}{4}$ d) $2{,}625 = \frac{21}{8}$ e) $1{,}875 = \frac{15}{8}$

15. Calculate: **Solution:**

a) $$\frac{\left(15 \cdot \frac{3}{20} + \frac{48}{55}\left(3 + \frac{7}{16}\right)\right) : \left(3 + \frac{1}{2}\right)}{\left(1 + \frac{9}{10}\right) : \left(1 + \frac{13}{25}\right) + \frac{3}{4} : 2} = \frac{12}{13}$$ b) $$\frac{\left(2 + \frac{1}{3}\right)^2 - \left(1 + \frac{5}{6}\right)^2 + \frac{1}{6}}{\left(4 + \frac{2}{5}\right)\frac{5}{12} + 2 + \frac{2}{3}} = \frac{1}{2}$$

16. Calculate: **Solution:**

a) $2{,}74 \cdot 1{,}08 = 2{,}9592$

b) $6{,}55 \cdot 1{,}58 = 10{,}349$

c) $\frac{40{,}11}{7} = 5{,}73$

d) $30 : 6{,}4 = 4{,}6875$

e) $\frac{1}{7} = 0,\overline{142857}$

10.2. Mathematical Operations and Transformations

1. Write the expressions without parentheses and simplify as much as possible.

a) $x^2(x^3 - 2x) - x^5$ **Solution:** $x^5 - 2x^3 - x^5 = -2x^3$

b) $x(x-1) - x^2$ **Solution:** $x^2 - x - x^2 = -x$

c) $(3x+2)^2$ **Solution:** $9x^2 + 12x + 4$

d) $3x(2-3x)$ **Solution:** $6x - 9x^2$

e) $x + 1 - 2(x-1)$ **Solution:** $x + 1 - 2x + 2 = 3 - x$

f) $2(x^2-1) - x^2$ **Solution:** $2x^2 - 2 - x^2 = x^2 - 2$

g) $1 - (1-3x)^2$ **Solution:** $1 - (1 - 6x + 9x^2) = 6x - 9x^2$

h) $2x^2(2x^2+1)$ **Solution:** $4x^4 + 2x^2$

i) $2(x+1) - 3(x+2)$ **Solution:** $2x + 2 - 3x - 6 = -x - 4$

j) $(x - x^3)x^2$ **Solution:** $x^3 - x^5$

2. Write the expressions without parentheses and simplify as much as possible:

a) $(9 + 4x - a) \cdot (-4)$ **Solution:** $-36 - 16x + 4a$

b) $(5a + 4b) \cdot (6x - 7y)$ **Solution:** $30ax - 35ay + 24bx - 28by$

c) $7x \cdot (2a - 3b - 4c) \cdot 2y$ **Solution:** $28axy - 42bxy - 56cxy$

d) $(4y + 6x) \cdot (3a - 5b) - (2y + 3x) \cdot (2a + 3b)$ **Solution:** $8ay - 26by + 12ax - 39bx$

e) $(15xy + 12bx) \cdot (a - c) - (5bx - 10x) \cdot (a - c)$
Solution: $15axy - 15cxy + 7abx - 7bcx + 10ax - 10cx$

f) $2n \cdot (3x + z) - (9x + 3z) \cdot (2n + 3) - 3x - z$ **Solution:** $-12nx - 4nz - 30x - 10z$

3. Skillfully combine the following : **Solution:**

   a) $23u-(14v-(8v+6u-3v-(43v-16u))-16u) = 23u-(14v-(22u-38v)-16u) = 23u-52v+38u = 61u-52v$

   b) $(3p-2q)^2-(3p+2q)^2 = -24pq$ with $(a-b)^2-(a+b)^2 = -2ab-2ab = -4ab$

   c) $(2a+b-c)x+(c-2a-b)y = (2a+b-c)(x-y)$

10.3. Calculating Fractions

1. Combine the fractions into a single fraction.

   a) $\frac{9}{4}\left(-\frac{8}{9}-\left(\frac{1}{3}+\frac{1}{6}\right)\right)$ **Solution:** $\frac{9}{4}\cdot\left(-\frac{16}{18}-\frac{6}{18}-\frac{3}{18}\right) = \frac{9}{4}\cdot\left(-\frac{25}{18}\right) = -\frac{25}{8}$

   b) $\left(\frac{3}{2}\cdot\frac{2}{3}-\left(\frac{1}{2}+\frac{1}{7}\right)\right)\cdot\frac{14}{15}$ **Solution:** $\frac{14}{15}\cdot\left(\frac{3}{2}\cdot\frac{2}{3}-\left(\frac{7}{14}+\frac{2}{14}\right)\right) = \frac{14}{15}\cdot\left(1-\frac{9}{14}\right) = \frac{14}{15}\cdot\frac{5}{14} = \frac{1}{3}$

   c) $\left(\frac{3}{5}-\frac{1}{25}\right):\left(1-\left(\frac{14}{50}-\frac{3}{100}\right)\right)$ **Solution:** $\left(\frac{15-1}{25}\right):\left(\frac{100-28+3}{100}\right) = \frac{14}{25}\cdot\frac{100}{75} = \frac{56}{75}$

   d) $\frac{7}{2}+\left(\left(\frac{3}{2}+\frac{15}{4}\right):\frac{1}{16}\right)$ **Solution:** $\frac{7}{2}+((\frac{6}{4}+\frac{15}{4})\cdot 16) = \frac{7}{2}+(\frac{21}{4}\cdot 16) = \frac{7}{2}+84 = \frac{175}{2}$

   e) $\frac{2}{3}:\left(\frac{4}{9}:\frac{3}{2}+\left(\frac{1}{2}-\frac{2}{3}:\frac{9}{4}\right)\right)$ **Solution:** $\frac{2}{3}:\left(\frac{4}{9}\cdot\frac{2}{3}+\frac{1}{2}-\frac{2}{3}\cdot\frac{4}{9}\right) = \frac{2}{3}:\left(\frac{8}{27}+\frac{1}{2}-\frac{8}{27}\right) = \frac{4}{3}$

   f) $\frac{49}{5}-\left(\frac{9}{10}:\frac{3}{8}\right):\left(\frac{1}{4}\cdot\left(\frac{12}{13}\cdot\frac{13}{12}\right)\right)$ **Solution:** $\frac{49}{5}-\left(\frac{9}{10}\cdot\frac{8}{3}\right)\cdot 4 = \frac{49}{5}-\frac{48}{5} = \frac{1}{5}$

2. Simplify as much as possible: **Solution:**

   a)
$$\frac{1}{y} = \frac{1}{x^2+1}+C \Rightarrow \frac{1}{y} = \frac{1+C(x^2+1)}{x^2+1} \Rightarrow y = \frac{x^2+1}{1+C(x^2+1)}$$

   b)
$$\frac{u-v}{v-u} = \frac{-(v-u)}{(v-u)} = -1$$

   c)
$$\frac{mt+ms-nt-ns}{mt-ms-nt+ns} = \frac{m(t+s)-n(t+s)}{m(t-s)-n(t-s)} = \frac{(m-n)(t+s)}{(m-n)(t-s)} = \frac{(t+s)}{(t-s)}$$

   d)
$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right):\frac{1}{abc} \Rightarrow \frac{bc+ac+ab}{abc}\cdot abc = bc+ac+ab$$

   e)
$$\frac{1}{x+\cfrac{1}{2x-\cfrac{2x^2}{1+x}}} = \frac{1}{x+\cfrac{1}{\cfrac{2x(1+x)-2x^2}{1+x}}} = \frac{1}{x+\cfrac{1+x}{2x}} = \frac{1}{\cfrac{2x^2+1+x}{2x}} = \frac{2x}{2x^2+x+1}$$

10.4. Exponent and Root Arithmetic, Logarithms

1. Simplify the expressions as much as possible.

   a) $a^{m-n+1}\cdot a^{m+n-8}$ **Solution:** $a^{m-n+1+m+n-8} = a^{2m-7}$

b) $x^{3n-6} \cdot 3x^{n-3m+2} \cdot 5x^{2-n}$ **Solution:** $15x^{3n-6+n-3m+2+2-n} = 15x^{3n-2-3m}$

c) $\sqrt[4]{a^{n+4}} : \sqrt{a^{n-4}}$ **Solution:** $\dfrac{(a^{n+4})^{0,25}}{(a^{n-4})^{0,5}} = \dfrac{a^{0,25n+1}}{a^{0,5n-2}} = a^{0,25n+1-0,5n+2} = a^{3-0,25n}$

d) $\sqrt{\sqrt[3]{x^8}}$ **Solution:** $\left((x^8)^{\frac{1}{3}}\right)^{\frac{1}{2}} = x^{\frac{8}{6}} = x^{\frac{4}{3}} = \sqrt[3]{x^4}$

e) $\left(\dfrac{x^{-4}y^{-5}}{a^{-1}b^3}\right)^2 \cdot \left(\dfrac{x^3a^{-2}}{y^2b^2}\right)^{-3}$ **Solution:**

$$\left(\frac{a}{x^4y^5b^3}\right)^2 \cdot \left(\frac{x^3}{y^2b^2a^2}\right)^{-3} = \frac{\frac{a^2}{x^8y^{10}b^6}}{\frac{x^9}{y^6b^6a^6}} = \frac{a^2}{x^8y^{10}b^6} \cdot \frac{y^6b^6a^6}{x^9} = \frac{a^8}{y^4x^{17}}$$

f) $\dfrac{45xa^3}{9yb^3} \cdot \dfrac{9y^n(a-1)^2}{30x^n(a+1)^2} : \dfrac{9y^{n-1}(1-a)^3}{24x^{n+1}(1+a)^2}$ **Solution:**

$$\begin{aligned}\frac{3xa^3y^n(a-1)^2}{2yb^3x^n(a+1)^2} : \frac{3y^{n-1}(1-a)^3}{8x^{n+1}(1+a)^2} &= \frac{3xa^3y^n(a-1)^2}{2yb^3x^n(a+1)^2} \cdot \frac{8x^{n+1}(1+a)^2}{3y^{n-1}(1-a)^3}\\ &= \frac{4a^3 \cdot xy^nx^{n+1}}{b^3(1-a)\cdot yx^ny^{n-1}} = \frac{4a^3 \cdot x^2}{b^3(1-a)} = \frac{4a^3x^2}{b^3-ab^3}\end{aligned}$$

2. Solve the equations for x.

a) $16x + 19 = 5(4+3x)$ **Solution:** $16x + 19 = 20 + 15x \Rightarrow x = 1$

b) $\dfrac{5}{2x} - \dfrac{3}{6x} = \dfrac{1}{3} - \dfrac{3}{x}$ **Solution:**
$\dfrac{15-3}{6x} = \dfrac{x-9}{3x} \Rightarrow \dfrac{12}{6x} = \dfrac{2x-18}{6x} \Leftrightarrow 12 = 2x - 18 \Leftrightarrow x = 15$

c) $5(2x+3) - 12(6-x) = 11(4x+7)$ **Solution:**
$10x + 15 - 72 + 12x = 44x + 77 \Rightarrow 22x = -134 \Rightarrow x = -\frac{134}{22} = -\frac{67}{11}$

d) $4x - 15(x-1) = 2(6-3x)$ **Solution:** $4x - 15x + 15 = 12 - 6x \Rightarrow 5x = 3 \Rightarrow x = \frac{3}{5}$

e) $\dfrac{12}{15x} + \dfrac{2}{3} = \dfrac{3}{5} + \dfrac{2}{3x}$ **Solution:** $\dfrac{12+10x}{15x} = \dfrac{9x+10}{15x} \Rightarrow 12 + 10x = 9x + 10 \Rightarrow x = -2$

f) $\dfrac{x+1}{3} - \dfrac{2x+5}{6} = \dfrac{3-4x}{2}$ **Solution:**
$\dfrac{2x+2-2x-5}{6} = \dfrac{9-12x}{6} \Rightarrow -3 = 9 - 12x \Rightarrow 12x = 12 \Rightarrow x = 1$

3. Solve the equations for x.

a) $-5 + 10x^2 + 5x + (x+2)(1-11x) = -21x - 3$ **Solution:**
$-5 + 10x^2 + 5x + (x - 11x^2 + 2 - 22x) = -21x - 3 \Rightarrow -3 - x^2 - 16x = -21x - 3$
$\Rightarrow 0 = x^2 - 5x = x(x-5) \Rightarrow x_1 = 0, x_2 = 5$

b) $-2x^2 + 3x - 3(-3x+7) = -6 + (3-x)(-7+2x)$ **Solution:**
$-2x^2 + 3x + 9x - 21 = -6 - 21 + 6x + 7x - 2x^2 \Rightarrow -2x^2 + 12x - 21 = -2x^2 + 13x - 27$
$\Rightarrow x = 6$

4. Solve the root equations for x. **Solution:**

a) $3\sqrt[3]{x-1} = \sqrt[3]{x+207} \Rightarrow 27(x-1) = x+207 \Rightarrow 27x-27 = x+207 \Rightarrow 26x = 234 \Rightarrow x = 9$

b) $\sqrt{x} = \sqrt{x+8}-2 \Rightarrow x = (x+8)-4\sqrt{x+8}+4 \Rightarrow -12 = -4\sqrt{x+8} \Rightarrow 3 = \sqrt{x+8} \Rightarrow 9 = x+8 \Rightarrow x = 1$

c) $\sqrt{x-1} = x-7 \Rightarrow x-1 = x^2-14x+49 \Rightarrow x^2-15x+50 = 0$
$\Rightarrow x_{1,2} = \frac{15\pm\sqrt{225-200}}{2} = \frac{15\pm5}{2} \Rightarrow x_1 = 10, x_2 = 5$
Checking with the initial equation shows: $x_2$can be dropped!

d) $\sqrt{x+2}\sqrt{x+7} = 6 \Rightarrow (x+2)(x+7) = 36 \Rightarrow x^2+7x+2x+14 = 36$
$\Rightarrow x^2+9x-22 = 0 \Rightarrow x_{1,2} = \frac{-9\pm\sqrt{81+88}}{2} = \frac{-9\pm13}{2} \Rightarrow x_1 = 2, x_2 = -11$
Checking with the initial equation shows: $x_2$can be dropped!

e) $\sqrt{2x+\sqrt{4x-3}} = 3 \Rightarrow 2x+\sqrt{4x-3} = 9 \Rightarrow \sqrt{4x-3} = 9-2x \Rightarrow$
$4x-3 = 81-36x+4x^2 \Rightarrow 4x^2-40x+84 = 0 \Rightarrow x^2-10x+21 = 0$
$\Rightarrow x_{1,2} = \frac{10\pm\sqrt{100-84}}{2} = \frac{10\pm4}{2} \Rightarrow x_1 = 7, x_2 = 3$
Checking with the initial equation shows: $x_1$can be dropped!

f) $3\sqrt{x+\sqrt{x-4}} = 6 \Rightarrow \sqrt{x+\sqrt{x-4}} = 2 \Rightarrow x+\sqrt{x-4} = 4 \Rightarrow \sqrt{x-4} = -x+4 \Rightarrow$
$x-4 = 16-8x+x^2 \Rightarrow$
$x^2-9x+20 = 0 \Rightarrow x_{1,2} = \frac{9\pm\sqrt{81-80}}{2} = \frac{9\pm1}{2}$
$\Rightarrow x_1 = 5, x_2 = 4$Checking with the initial equation shows: $x_1$can be dropped!

g) $\sqrt{4x+\sqrt{x+3}} = \sqrt{5x+1} \Rightarrow 4x+\sqrt{x+3} = 5x+1 \Rightarrow \sqrt{x+3} = x+1$
$\Rightarrow x+3 = x^2+2x+1 \Rightarrow x^2+x-2 = 0 \Rightarrow x_{1,2} = \frac{-1\pm\sqrt{1+8}}{2} = \frac{-1\pm3}{2}$
$\Rightarrow x_1 = -2, x_2 = 1$Checking with the initial equation shows: $x_1$can be dropped!

h) $\sqrt[4]{x+2} = \sqrt[8]{4x+8} \Rightarrow x+2 = \sqrt{4x+8} \Rightarrow x^2+4x+4 = 4x+8 \Rightarrow x^2-4 = 0 \Rightarrow x_1 = 2, x_2 = -2$

5. Calculate: **Solution:**

a) $\left((-2)^{-2}\right)^3 = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$

b) $\left((-2)^3\right)^{-2} = (-2)^{-6} = \frac{1}{64}$

c) $(-2^3)^2 = (-8)^2 = 64$

d) $(-2^3)^{-3} = -\frac{1}{512}$

e) $(-x^3)(-x^2)(-x)^4 = (-1)(-1)(+1)x^{(3+2+4)} = x^9$

f) $\frac{(-2x)^{-2m}}{-2x^{-2m+1}} = \frac{(-2)^{-2m}x^{-2m}}{-2\cdot x^{-2m}\cdot x} = \frac{2^{-2m}}{-2\cdot x} = -\frac{1}{2^{2m+1}\cdot x}$

g) $\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^3} = \frac{x^2-2x+1}{x^3} = \frac{(x-1)^2}{x^3}$

6. Solve the equations for x.

a) $3^{x+1} = 5$ **Solution:** $\ln(3^{x+1}) = \ln 5 \Rightarrow x+1 = \frac{\ln 5}{\ln 3} \Rightarrow x = \frac{\ln 5}{\ln 3} - 1$

b) $5^{2x-1} = 12$ **Solution:** $\ln(5^{2x-1}) = \ln 12 \Rightarrow 2x-1 = \frac{\ln 12}{\ln 5} \Rightarrow x = (\frac{\ln 12}{\ln 5}+1)\cdot\frac{1}{2}$

c) $5 \cdot 3^{x+2} = 72$ **Solution:**

$$\ln\left(3^{x+2}\right) = \ln\left(\frac{72}{5}\right) \Rightarrow (x+2)\ln 3 = \ln 72 - \ln 5$$
$$\Rightarrow x+2 = \frac{\ln 72 - \ln 5}{\ln 3} \Rightarrow x = \frac{\ln 72 - \ln 5}{\ln 3} - 2$$

d) $15 \cdot e^{2x} = 66$
Solution: $\ln e^{2x} = \ln\left(\frac{66}{15}\right) \Rightarrow 2x = \ln\left(\frac{22}{5}\right) \Rightarrow 2x = \ln 22 - \ln 5 \Rightarrow x = \frac{\ln 22 - \ln 5}{2}$

e) $1 - e^{5x} = 0{,}3$
Solution: $e^{5x} = 0{,}7 \Rightarrow \ln e^{5x} = \ln 0{,}7 \Rightarrow 5x = \ln 0{,}7 \Rightarrow x = \frac{\ln 0{,}7}{5}$

f) $5^{3x} = 7^{2x}$ **Solution:** $\ln 5^{3x} = \ln 7^{2x} \Rightarrow 3x \cdot \ln 5 = 2x \cdot \ln 7 \Rightarrow x = 0$

g) $10^{7-x} = 6^{2x+1}$ **Solution:**

$$\ln 10^{7-x} = \ln 6^{2x+1} \Rightarrow (7-x)\ln 10 = (2x+1)\ln 6$$
$$\Rightarrow 7\ln 10 - x\ln 10 = 2x\ln 6 + \ln 6 \Rightarrow 7\ln 10 - \ln 6 = 2x\ln 6 + x\ln 10$$
$$\Rightarrow 7\ln 10 - \ln 6 = x\,(2\ln 6 + \ln 10) \Rightarrow x = \frac{7\ln 10 - \ln 6}{2\ln 6 + \ln 10}$$

h) $3^x \cdot 4^{x+1} = 5^{x+2}$ **Solution:**

$$\ln\left(3^x \cdot 4^{x+1}\right) = \ln\left(5^{x+2}\right) \Rightarrow x\ln 3 + (x+1)\ln 4 = (x+2)\ln 5$$
$$\Rightarrow x\ln 3 + x\ln 4 + \ln 4 = x\ln 5 + 2\ln 5 \Rightarrow x\ln 3 + x\ln 4 - x\ln 5 = 2\ln 5 - \ln 4$$
$$\Rightarrow x\,(\ln 3 + \ln 4 - \ln 5) = 2\ln 5 - \ln 4 \Rightarrow x = \frac{2\ln 5 - \ln 4}{\ln 3 + \ln 4 - \ln 5}$$

i) $\log_x 81 = \frac{1}{4}$ **Solution:** $x^{\frac{1}{4}} = 81 \Rightarrow x = 81^4$

j) $\log_7 x = -2$ **Solution:** $x = 7^{-2} \Rightarrow x = \frac{1}{7^2} \Rightarrow x = \frac{1}{49}$

k) $\log_2 x = -8$ **Solution:** $x = 2^{-8} \Rightarrow x = \frac{1}{2^8} \Rightarrow x = \frac{1}{256}$

l) $80 = 16 \cdot 4^x$ **Solution:** $4^x = 5 \Rightarrow x = \log_4 5 = \frac{\ln 5}{\ln 4}$

m) $1000 = 0{,}5^x$ **Solution:** $x = \log_{0{,}5} 1000 = \frac{\ln 1000}{\ln 0{,}5}$

n) $2^x = \frac{1}{\sqrt{\sqrt{2}}}$ **Solution:** $2^x = \frac{1}{2^{\frac{1}{2}\frac{1}{2}}} = \frac{1}{2^{\frac{1}{4}}} \Rightarrow 2^x = 2^{-\frac{1}{4}} \Rightarrow x = -\frac{1}{4}$

o) $4^x = 3{,}2$ **Solution:** $x = \log_4 3{,}2 = \frac{\ln 3{,}2}{\ln 4}$

p) $2^{-x} = -2^x$ **Solution:** $2^{-x} = -2^x \Rightarrow \frac{1}{2^x} \neq -2^x \forall x \in \mathbb{R}$ This means the equation is not solvable!

7. Simplify the following: **Solution:**

a) Since $2k-2$ is even, then $(a-b)^{2k-2} = (b-a)^{2k-2}$. Therefore

$$5(b-a)^{2k-2} \cdot \frac{9}{5}(b-a)^{7-2k} \cdot \frac{2}{3}(b-a)^{2k-5} = 6(b-a)^{2k} = 6(a-b)^{2k}$$

b)

$$\frac{a-b}{\sqrt{a}-\sqrt{b}}-\frac{a-b}{\sqrt{a}+\sqrt{b}}=\frac{(a-b)(\sqrt{a}+\sqrt{b})-(a-b)(\sqrt{a}-\sqrt{b})}{a-b}=2\sqrt{b}$$

c)

$$\sqrt{\frac{a}{b}\sqrt{\frac{b}{a}\sqrt{\frac{a}{b}}}}=\left(\frac{a}{b}\left(\frac{b}{a}\left(\frac{a}{b}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}=\left(\frac{a}{b}\right)^{\frac{1}{2}}\left(\frac{a}{b}\right)^{-\frac{1}{4}}\left(\frac{a}{b}\right)^{\frac{1}{8}}=\left(\frac{a}{b}\right)^{\frac{3}{8}}$$

8. Calculate or split up the following: **Solution:**

   a) $\log_a \frac{1}{a}=\log_a a^{-1}=-1$

   b) $\log_a a=1$

   c) $\log_a a^n=n$

   d) $\ln\left(\sqrt[4]{a^3}\right)=\ln\left(a^{\frac{3}{4}}\right)=\frac{3}{4}\ln a$

   e) $\ln\left(\frac{x^2-y^2}{x^2+y^2}\right)=\ln(x^2-y^2)-\ln(x^2+y^2)=\ln((x-y)(x+y))-\ln(x^2+y^2)$
   $=\ln(x-y)+\ln(x+y)-\ln(x^2+y^2)$

   f) $\ln e^e=e$

   g) $\ln\sqrt{e}^e=\ln e^{\frac{e}{2}}=\frac{e}{2}$

   h) $\ln e^{\sqrt{e}}=\sqrt{e}$

   i) $\ln\left(\frac{u-v}{e(v-u)}\right)^2=\ln\left(\left(\frac{1}{e^2}\right)\frac{(u-v)^2}{(v-u)^2}\right)=\ln\left(e^{-2}\cdot 1\right)=-2$

   j) $\log_2 x=3\Rightarrow x=2^3=8$

   k) $\log_x 2197=3\Rightarrow 2197=x^3\Rightarrow x=\sqrt[3]{2197}=13$Note: The equation $2197=x^3$ can be solved simply in your head by applying some skillful testing. 2197 has four places, so x must have two places. Since $10^3=1000$ and $20^3=8000$, the solution must lie close to10 . Since 2197 is uneven, x cannot be even. Eleven is certainly still a bit too small, calculating roughly:

   $$11^3=121\cdot 11<1400$$

   The next number to test is therefore $x=13$. Thus, e.g.: $13^3=169\cdot 13=1690+3\cdot 100+(3\cdot 70-3)=1990+210-3$, that fits.

10.5. Numbers and Sets

1. Assign the following numbers to the appropriated fundamental sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$: **Solution:**
   $12\in\mathbb{N}$
   $12,\ -8\in\mathbb{Z}$
   $12,\ -8,\ \frac{2}{3},\ 1{,}41\in\mathbb{Q}$
   $12,\ -8,\ \frac{2}{3},\ 1{,}41,\ \sqrt{7},\ \pi\in\mathbb{R}$

2. Given three sets: $M_1 = \{a, b, c, d, e\}$. $M_2 = \{e, f, g, h, i\}$ and $M_3 = \{a, c, e, g, i\}$. Determine the sets $M_1 \cap M_2$. $M_1 \cup M_2$. $M_1 \cap M_3$. $M_1 \cup M_3$. $M_2 \cap M_3$ and $M_2 \cup M_3$. In addition, also determine $M_1 \backslash M_2$. $M_2 \backslash M_1$. $M_2 \backslash M_3$ and $M_1 \cap M_2 \cap M_3$ as well as$M_1 \cup M_2 \cup M_3$.
Solution:
$M_1 \cap M_2 = \{e\}$, $M_1 \cup M_2 = \{a, b, c, d, e, f, g, h, i\}$, $M_1 \cap M_3 = \{a, c, e\}$, $M_1 \cup M_3 = \{a, b, c, d, e, g, i\}$, $M_2 \cap M_3 = \{e, g, i\}$ and $M_2 \cup M_3 = \{a, c, e, f, g, h, i\}$.
Further, $M_1 \backslash M_2 = \{a, b, c, d\}$, $M_2 \backslash M_1 = \{f, g, h, i\}$, $M_2 \backslash M_3 = \{f, h\}$ and $M_1 \cap M_2 \cap M_3 = \{e\}$ as well as$M_1 \cup M_2 \cup M_3 = \{a, b, c, d, e, f, g, h, i\}$.

3. Given the following subsets of the real numbers:

$$A = \{x \in \mathbb{R} \mid -5 < x < 2\}, B = \{x \in \mathbb{R} \mid 4 > x \geqslant 0\},$$
$$C = \left\{x \in \mathbb{R} \mid x^2 \leqslant 9\right\}, D = \left\{x \in \mathbb{R} \mid x^2 > 1\right\}$$

Determine the following sets. You can first sketch the sets on the number line if you are having difficulties.

a) $A \cup B$ **Solution:** $\{x \in \mathbb{R} \mid -5 < x < 4\} = \,]-5; 4\,[$

b) $A \backslash B$ **Solution:** $\{x \in \mathbb{R} \mid -5 < x < 0\} = \,]-5; 0\,[$

c) $A \cap B$ **Solution:** $\{x \in \mathbb{R} \mid 0 \leqslant x < 2\} = [\,0; 2\,[$

d) $C \cap D$ **Solution:** $\{x \in \mathbb{R} \mid -3 \leqslant x < -1 \vee 1 < x \leqslant 3\} = [\,-3; -1\,[\, \cup \,]\,1; 3\,]$

e) $(\mathbb{R} \backslash D) \cup B$ **Solution:** $\{x \in \mathbb{R} \mid -1 \leqslant x < 4\} = [\,-1; 4\,[$

f) $C \backslash (A \cap B)$ **Solution:** $\{x \in \mathbb{R} \mid -3 \leqslant x < 0 \vee 2 \leqslant x \leqslant 3\} = [\,-3; 0\,[\, \cup [2; 3]$

g) $(A \cap B) \cup (C \cap D)$
Solution: $\{x \in \mathbb{R} \mid -3 \leqslant x < -1 \vee 0 \leqslant x \leqslant 3\} = [\,-3; -1\,[\, \cup [0; 3]$

4. Solve the following for x and determine the solution set.

a) $4x + 17 > -x - 3$ **Solution:** $L = \{x | x > -4\} = \,]-4, \infty\,[$

b) $-4 \cdot (3x - 2) < 6 \cdot (1 - 2x)$ **Solution:** $L = \{\ \}$

10.6. Quantities and Units

1. Which of the following units are coherent or incoherent, respectively?
Solution:

a) Newton is coherent

b) Kilonewton is incoherent

c) Millinewton is incoherent

d) Kilogram is coherent

e) Kilometer is incoherent

f) Nanosecond is incoherent

2. Convert all given distances into kilometers. **Solution:**

a) 12 m = 0.012 km

b) 3.4 m = 0.0034 km

c) 0.5 m = 0.0005 km

d) 100 mm = 0.0001 km

e) 120 dm =0.012 km

f) 150 mm = 0.00015 km

g) 5000 m = 5 km

h) 200000 μm = 0.0002 km

3. Convert all given distances into millimeters. **Solution:**

   a) 12 km = $12 \cdot 10^6$ mm
   b) 120 m = 120000 mm
   c) 0.5 m = 500 mm
   d) 100 cm = 1000 mm
   e) 50 km = $50 \cdot 10^6$ mm
   f) 500 m = 500000 mm
   g) 800 m = 800000 mm
   h) 400 cm = 4000 mm

4. Maritime travel uses the sea mile as the unit of length and the knot as the unit for velocity, where1 sea mile = 1sm = 1852m and 1 knot = 1kn= 1 sm/h. Convert the velocity 72 kn into the units km/h and m/s.
   Solution: 72kn = 133,344 km/h = 37,04 m/s

5. How many minutes and seconds are missing from 7 min 12 sec in order to round out a complete hour?
   Solution: A total of 52 min and 48s are missing to complete a full hour.

6. An indefatigable tortoise completes a trek of 100.8 km in a time of two weeks. Calculate the tortoise's velocity in km/h and in cm/s.
   Solution: 0.3km/h ; 8.3cm/s

7. On a road map with scale 1: 250000 the distance to be driven is 12.3 cm long. How much time is will be required to drive this distance at an average velocity of 50 km/h?
   Solution: 0.615h

8. A model railroad has a scale of 1: 125.

   a) Which distance must the little train travel in an hour if the actual velocity should correspond to 80 km/h?
      Solution: 640m

   b) Which distance does the little train travel in one minute and which in five minutes?
      Solution: 10,667 m, 53,333 m

   c) Which distance would the actual train have travelled if the little train has travelled 300 cm in 10 s?
      Solution: 375 m

   d) Which velocity in km/h would the actual train then have reached?
      Solution: 135 km/h

9. Area calculation

   a) The old Federal Republic of Germany had an area of 248,800 km^2 and a population of 61.6 million. Calculate the population density (number of inhabitants per km^2)
      Solution: 247.6 inhabitants perkm^2

   b) Canada has an area of 9,976,000 km^2 and a population of 24 million. What is the ratio of the population density of the old Federal Republic of Germany to that of Canada's?
      Solution: 1:103

   c) How often does the area of the old Federal Republic of Germany fit into the area of Canada?
      Solution: approximately 40.0 times

10. The speed of sound is approximately 333 m/s. What is the speed of an airplane in

a) km/h and m/min at twice the speed of sound?
Solution: 2397.6 km/h; 39,960 m/min

b) km/h and m/min at four times the speed of sound?
Solution: 4795.2 km/h: 79,920 m/min

10.7. Cross-Multiplication

1. 12 pieces of work produce 3.3 kg of shavings. How many kg of shavings result from 100 pieces of work? **Solution:** 27 5 kg

2. A container holds 450 l of water and is filled from an open tap in 12 minutes. How many liters of water were in the container after 7 minutes? **Solution:** 262.5 ℓ

3. A commercial airliner with a velocity of 850 km/h travels a certain distance in 55 minutes. How much time does a new type of airplane which can travel at 950 km/h require for the same distance.
Solution: The new type of airplane needs about 49 minutes to complete the distance.

4. Find the error in the calculation. The final price for an automobile is €23,800. What is the price before taxes?

$$\begin{array}{rr} 100 & 23800 \\ :100\downarrow & :100\downarrow \\ 1 & 238 \\ \cdot 19\downarrow & \cdot 19\downarrow \\ 19 & 4522 \end{array}$$

The price before taxes is € 23,800 - € 4522 = € 19,278.

Solution: The 19 % value added tax are already contained in the final price, i.e., 23800$\widehat{=}$119%. The price before taxes is actually 20000 Euro.

10.8. Systems of Linear Equations

Determine the solutions to the following systems of linear equations.

1. $5x - 2y = 24$
$x + 3y = -2$
Solution: $x = 4,\ y = -2$

2. $2x - 2y = 2$
$-x + y = -2$
Solution: No solution.

3. $y = 4x - 7$
$-8x + 2y = -14$
Solution: $y = 4x - 7$ where $x \in \mathbb{R}$

4. $b + 3c = 10$
$5a - 3b + c = 5$
$-2b + 2c = -4$
Solution: $a = 3,\ b = 4,\ c = 2$

5. $x + 2y + z = 9$
$-2x - y + 5z = 5$
$x - y + 3z = 4$
Solution: $x = 1,\ y = 3\ z = 2$

6. $4x + y + 7z = 12$
$5x + 10z = 5$
$-x - 2y = -2$
Solution: No solution.

7. $x - y + z = -2$
$4x + 2y + z = -5$
$6x + 3z = -9$
Solution: $x = t,\ y = -t{-}1,\ z = -2t{-}3$ where $t \in \mathbb{R}$

10.9. For Experienced Students

1. Do the arithmetic in your head: Begin with the given number and follow the arithmetic directions from top to bottom. Try to be skillful in your calculations. The degree of difficulty increases from left to right. If you are experienced, you should require less than a minute per column.

Simple		**Intermediate**		**Difficult**	
40		57		98	
take $\frac{1}{4}$	$=10$	$\cdot 3$	$= 171$	$+524$	$= 622$
square it	$=100$	$+211$	$=382$	$+\frac{1}{2}$ of the value	$= 933$
$+24$	$= 124$	take half	$=191$	$/3$	$= 311$
$/4$	$= 31$	-52	$= 139$	-243	$= 68$
$\cdot 2$	$= 62$	double it	$=278$	$\cdot 5$	$= 340$
-30	$=32$	$+122$	$=400$	$/20$	$=17$
$/4$	$=8$	$/20$	$=20$	square it	$= 289$
$\cdot 7$	$= 56$	$+\frac{3}{10}$ of the value	$=26$	$+473$	$= 762$
-22	$= 34$	$/13$	$=2$	$+\frac{2}{3}$ of the value	$= 1270$
34		2		1270	

2. Determine the solution set of the following equations.

   a) $\frac{9}{x-8} = x$ **Solution:** $L = \{-1;\ 9\}$

   b) $\frac{5}{x-2} - 3 = \frac{2x-4}{5}$ **Solution:** $L = \{-\frac{7}{4} \pm \frac{5}{4}\sqrt{17}\}$

   c) $\frac{2x+1}{3} + \frac{10}{2x+1} = 4$ **Solution:** $L = \{\frac{5}{2} \pm \frac{1}{2}\sqrt{6}\}$

   d) $\frac{2x}{x-4} + \frac{3x}{x+4} = \frac{4(x^2-x+4)}{x^2-16}$ **Solution:** $L = \{\ \}$, since the calculated solutions are zeros of the denominator.

3. Determine the solution set dependent on p.
   $5 \cdot (3 + x) > p \cdot (1 - x) + 3 \cdot (x + 5)$
   Solution: $p = -2:\ L = \mathbb{R};\ p > -2:\ L = \left\{x | x > \frac{p}{p+2}\right\};\ p < -2:\ L = \left\{x | x < \frac{p}{p+2}\right\}$

4. A farmer wants to dry freshly harvested fruit. He spreads approximately 100 kilograms of it out on a large tarpaulin and lets the sun go to work. Initially, the water content is at 99 percent.
   A few days later the water content has been reduced to 98 percent. How much does the fruit weigh now - including all the water it still contains?
   Solution: Initially, the dry mass accounted for 1 percent of the total weight, i.e. 1 kg. The dry mass remains constant throughout the drying process. That means that at the next weighing this corresponds to $100 - 98 = 2$ percent of the total mass. Thus the fruit now weighs only 50 kg.

5. Determine the solutions to the following systems of equations. **Solution:**

   a) $-4x + 7y - 5z = 4$
   $6x + 3z = -8$
   $-4x - 8y - 6z = -8$
   $x = -\frac{7}{3},\ y = \frac{2}{3},\ z = 2$

   b) $2a + 6b - 3c = -6$
   $4a + 3b + 3c = 6$
   $4a - 3b + 9c = 18$
   $a = -1{,}5t + 3, t \in \mathbb{R}$
   $b = t - 2,\ c = t$

   c) $2x - 3y - z = 4$
   $x + 2y + 3z = 1$
   $3x - 8y - 5z = 5$
   No solution

11. Functions

11.1. Linear Functions

1. Determine the equation of the form $f(x) = ax + b$ based on the given information.

   a) $a = 3$ and the point $P(2; 16)$ lies on the line. **Solution:** $f(x) = 3x + 10$.

   b) The line goes through the point $Q(-1; -2)$ and is parallel to the line with equation $g(x) = -2x + 1$. **Solution:** $f(x) = -2x - 4$.

   c) It is $b = -1\ 5$ and $f(2{,}5) = 16$. **Solution:** $f(x) = 7x - 1\ 5$.

   d) It is $f(3) = 4$ and $f(-9) = 8$. **Solution:** $f(x) = -\frac{1}{3}x + 5$.

2. A company manufactures chairs. The production costs consist of fixed costs (rent, machines, ...) amounting to 20,000 Euros and variable costs (materials, labor, ...) amounting to 30 Euros for each individual chair. Chairs are sold for 80 Euros apiece.

   a) Determine the function describing the profit (income minus costs) as a function of the number of chairs. **Solution:** $f(x) = 80 \cdot x - 30 \cdot x - 20{,}000 \Rightarrow f(x) = 50x - 20{,}000$.

   b) How high is the profit if 3000 chairs are sold?
   Solution: $f(3000) = 130{,}000$.

   c) As of what number of sold chairs does the company begin to make a profit?
   Solution: $x = 400$.

11.2. Quadratic Functions

1. Solve for the roots and complete the square in order to determine the vertex form for function $f(x)$.

   a) $f(x) = x^2 - 4x + 1$ **Solution:** $x_{1/2} = 2 \pm \sqrt{3}$, $f(x) = (x-2)^2 - 3$

   b) $f(x) = x^2 + 3x - 4$ **Solution:** $x_1 = -4, \quad x_2 = 1$, $f(x) = (x - \frac{3}{2})^2 - \frac{25}{4}$

   c) $f(x) = -2x^2 + 4x - 8$ **Solution:** No real roots, $f(x) = -2(x-1)^2 - 6$

2. The quadratic function $f(x) = ax^2 + bx + c$ has the roots -1 and 3. For the following, determine the parameters a, b and c for the following valued of f at point $x = 0$:

   Solution: The following applies for all cases: $f(-1) = 0 \Rightarrow a - b + c = 0$ and $f(3) = 0 \Rightarrow 9a + 3b + c = 0$.

   a) $f(0) = 3 \Rightarrow c = 3$ The system of linear equations gives $a = -1$ and $b = 2$.

   b) $f(0) = -3 \Rightarrow c = -3$ The system of linear equations gives $a = 1$ and $b = -2$.

   c) $f(0) = -6 \Rightarrow c = -6$ The system of linear equations gives $a = 2$ and $b = -4$.

   d) $f(0) = 1\ 5 \Rightarrow c = 1\ 5$ The system of linear equations gives $a = -0{,}5$ and $b = 1$.

   Alternative approach: $f(x) = a(x+1)(x-3)$.

11.3. Rational Functions

1. Solve the following equations for x. Make certain that it is clear to you for which function you have determined the roots by doing so. **Solution:**

   a) $x^2 + x = 0 \Leftrightarrow x^2 = -x \Rightarrow x_1 = -1,\ x_2 = 0$

   b) $(x-3)^2 = 16 \Leftrightarrow x - 3 = \pm\sqrt{16} \Leftrightarrow x = 3 \pm 4 \Rightarrow x_1 = 7,\ x_2 = -1$

   c) $\dfrac{1}{x^2} - \dfrac{1}{x} - 2 = 0 \Leftrightarrow \dfrac{1-x}{x^2} - 2 = 0 \Leftrightarrow 1 - x - 2x^2 = 0 \Rightarrow x_1 = \frac{1}{2},\ x_2 = -1$

   d) $\dfrac{1-x}{1+x} = x \Leftrightarrow 1 - x = x + x^2 \Leftrightarrow 0 = x^2 + 2x - 1 \Rightarrow x_1 = -1 + \sqrt{2},\ x_2 = -1 - \sqrt{2}$

   e) The solution applies only for $b \neq 0$. In case $b = 0$, then $a = -c$ must apply and x is arbitrary.

   $$\begin{aligned} &a - (a-b)\,x = (b-a)\,x - (c + bx) \\ \Leftrightarrow\ &a - ax + bx \Leftrightarrow bx = -(a+c) = bx - ax - c - bx \Leftrightarrow x = -\frac{a+c}{b} \end{aligned}$$

   f) The solution can be recognized intuitively: $\sqrt{2x^2 - 1} = -x \Leftrightarrow x = -1$. If instead the equation is conventionally squared, it is necessary to use trial and error to find the solution with the correct algebraic sign: $2x^2 - 1 = x^2 \Leftrightarrow x = \pm 1$. The solution $x = 1$ is not possible since the square root expression (left hand side) does not allow a negative solution (right hand side).

2. Determine the roots of the functions.

   a) $f(x) = x^3 - 2x^2 - 11x + 12$ **Solution:** $x_1 = -3,\ x_2 = 1,\ x_3 = 4$

   b) $f(x) = 2x^3 - 6x^2 + 8$ **Solution:** $x_1 = -1$, double root at $x_2 = 2$

   c) $f(x) = -\frac{1}{2}x^3 - 6x^2 - 4{,}5x + 11$ **Solution:** $x_1 = -11,\ x_2 = -2,\ x_3 = 1$

   d) $f(x) = x^4 - 4x^3 - 13x^2 + 4x + 12$ **Solution:** $x_1 = -2,\ x_2 = -1,\ x_3 = 1,\ x_4 = 6$

   e) $f(x) = x^3 - 3x^2 - 2x - 40$ **Solution:** $x_1 = 5$

   f) $f(x) = 20x^3 - 120x^2 + 220x - 120$ **Solution:** $x_1 = 1,\ x_2 = 2,\ x_3 = 3$

11.4. Trigonometric Functions

1. Determine the angle in radians as a multiple of π .

   a) $180°, 270°, 315°, 135°$ **Solution:** $\pi,\ \frac{3}{2}\pi,\ \frac{7}{4}\pi,\ \frac{3}{4}\pi$

   b) $2°, 10°, 100°, 60°$ **Solution:** $\frac{1}{90}\pi,\ \frac{1}{18}\pi,\ \frac{5}{9}\pi,\ \frac{\pi}{3}$

2. Determine the angle in degrees.

   a) $3\pi,\ \frac{3}{10}\pi,\ \frac{\pi}{2},\ \frac{2}{3}\pi$ **Solution:** $540°, 54°, 90°, 120°$

   b) $\frac{\pi}{6},\ \frac{\pi}{18},\ \frac{5}{18}\pi,\ \frac{5}{4}\pi$ **Solution:** $30°, 10°, 50°, 225°$

3. Solve for the exact value of the function.

a) $\sin\left(\frac{3}{2}\pi\right)$ **Solution:** $\sin\left(\frac{3}{2}\pi\right) = -1$

b) $\sin\left(-\frac{7}{4}\pi\right)$ **Solution:** $\sin\left(-\frac{7}{4}\pi\right) = \frac{1}{2}\sqrt{2}$

c) $\cos\left(\frac{2}{3}\pi\right)$ **Solution:** $\cos\left(\frac{2}{3}\pi\right) = -\frac{1}{2}$

d) $\cos(-3\pi)$ **Solution:** $\cos(-3\pi) = -1$

4. Solve for the period and amplitude of f. In addition, without calculating, determine the coordinates of one maximum H and one minimum T of the graph of f.

   a) $f(x) = 2\sin(2x)$ **Solution:** $p = \pi,\ a = 2,\ H = (\pi/4; 2),\ T = (3/4\pi; -2)$

   b) $f(x) = \sin(\pi(x-1)) + 1$ **Solution:** $p = 2,\ a = 1,\ H = (1/2; 0),\ T = (3/2; 2)$

   c) $f(x) = \cos\left(\frac{\pi}{2}x\right) - 1$ **Solution:** $p = 4,\ a = 1,\ H = (0; 0),\ T = (2; -2)$

   d) $f(x) = -1\ 5\cos(x+\pi)$ **Solution:** $p = 2\pi,\ a = 1\ 5,\ H = (2\pi; 1\ 5),\ T(\pi; -1\ 5)$

5. Solve. **Solution:**

   a) $\sin(2x) + 7 = 8,\ x \in [0; 2\pi] \Rightarrow \sin(2x) = 1 \Rightarrow 2x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{4} \Rightarrow L = \left\{\frac{\pi}{4};\ \frac{5}{4}\pi\right\}$
   The remaining solution is obtained by means of the periodicity of the function ($p = \pi$) on the given interval under consideration.

   b) $\cos\left(\frac{\pi}{4}x\right) - 1 = 0,\ x \in [0; 10] \Rightarrow \cos\left(\frac{\pi}{4}x\right) = 1 \Rightarrow \frac{\pi}{4}x = 0 \Rightarrow x = 0 \Rightarrow L = \{0;\ 8\}$
   The remaining solution is obtained by means of the periodicity of the function ($p = 8$) on the given interval under consideration.

11.5. The Inverse Function

1. Determine the domain $D_f \subseteq \mathbb{R}$ of the function $f : D_f \to \mathbb{R}$ defined by $f(x)$ so that it can be unambiguously inverted. Determine the inverse function and its domain.

   a) $f(x) = \dfrac{1}{x-1}$ **Solution:** $D_f = \mathbb{R}\backslash\{1\}$ and $f^{-1}(x) = \dfrac{1}{x} + 1,\ D_{f^{-1}} = \mathbb{R}\backslash\{0\}$

   b) $f(x) = \dfrac{x+3}{5x-7}$ **Solution:** $D_f = \mathbb{R}\backslash\left\{\dfrac{7}{5}\right\}$ and $f^{-1}(x) = \dfrac{7x+3}{5x-1},\ D_{f^{-1}} = \mathbb{R}\backslash\left\{\dfrac{1}{5}\right\}$

   c) $f(x) = \sqrt{2x+6}$ **Solution:** $D_f = \{x \in \mathbb{R},\ x \geqslant -3\}$ and $f^{-1}(x) = \dfrac{x^2}{2} - 3,\ D_{f^{-1}} = \mathbb{R}_0^+$

2. Determine the inverse function of each of the following functions $f : (0, \infty) \to \mathbb{R}$, each in turn defined by $f(x)$. Note: $x > 0$ applies in each case.

   a) $f(x) = 3x^2 + x$ **Solution:** $f^{-1}(x) = -\frac{1}{6} + \sqrt{\frac{1}{36} + \frac{x}{3}},\ ,\ D_{f^{-1}} = \left(-\frac{1}{12}, \infty\right)$

   b) $f(x) = (x^2+1)^{-1}$ **Solution:** $f^{-1}(x) = \sqrt{\frac{1}{x} - 1},\ D_{f^{-1}} = (0{,}1]$

   c) $f(x) = \sqrt{-2x}$ **Solution:** The function $f(x)$ is not defined for $x > 0$. Therefore, no inverse function exists. If $x < 0$, then the inverse function is given by : $f^{-1}(x) = -\frac{1}{2}x^2,\ D_{f^{-1}} = (0, \infty)$

11.6. Root Functions

Solve for the zeroes of the following functions.

1. $f(x) = x^4 - 625$ **Solution:** $x_{1,2} = \pm 5$
2. $f(x) = 2x^3 + 0{,}25$ **Solution:** $x = -0{,}5$
3. $f(x) = \dfrac{x^4}{8} + 2$ **Solution:** no solution
4. $f(x) = (x-3)^5 + \frac{1}{32}$ **Solution:** $x = \frac{5}{2}$
5. $f(x) = \sqrt[3]{x} + 8$ **Solution:** no solution
6. $f(x) = \sqrt[5]{x-1} - 2$ **Solution:** $x = 33$
7. $f(x) = x^{-1} - 5\sqrt{x^3}$ **Solution:** $x = \dfrac{1}{\sqrt[5]{25}}$

11.7. Exponential Functions and Logarithms

1. Solve exactly for the roots of the following functions.
   a) $f(x) = e^x - 5$ **Solution:** $x = \ln 5$
   b) $f(x) = -3e^x + 2$ **Solution:** $x = \ln 2 - \ln 3$
   c) $f(x) = e^{-x} + 4$ **Solution:** No solution, since $e^{-x} > 0$ for all $x \in \mathbb{R}$
   d) $f(x) = \frac{5}{2}e^{-2x} - \frac{5}{2}e$ **Solution:** $x = -\frac{1}{2}$
2. Determine the exponential function of form $f(x) = a \cdot q^x$ for the function whose plot goes through the points $P = (0; 1\ 5)$ and $Q = (2; 6)$.
   Solution: $f(0) = 1\ 5 \Rightarrow a = 1\ 5,\ f(2) = 6 \Rightarrow 6 = 1\ 5 \cdot q^2 \Rightarrow q = 2 \Rightarrow f(x) = 1\ 5 \cdot 2^x$
3. A nutrient solution initially contains 50,000 bacteria at the beginning of an experiment. The number of bacteria increases by 10% daily.
   a) Determine the corresponding growth function. **Solution:**
   $f(t) = 50{,}000 \cdot 1{,}1^t = 50{,}000 \cdot e^{\ln 1{,}1 \cdot t}$, where t is the number of days.
   b) How many bacteria are in the nutrient solution after five days? **Solution:** $f(5) = 80{,}526$
   c) Determine the length of time it takes for the number of bacteria to double. **Solution:**
   $2 \cdot 50{,}000 = 50{,}000 \cdot e^{\ln 1{,}1 \cdot t} \Rightarrow t = \frac{\ln 2}{\ln 1{,}1} \approx 7\ 27$ days
   d) After how much time has the number of bacteria increased by a factor of ten? **Solution:**
   $10 \cdot 50{,}000 = 50{,}000 \cdot e^{\ln 1{,}1 \cdot t} \Rightarrow t = \frac{\ln 10}{\ln 1{,}1} \approx 24\ 16$ This means that there has been a tenfold increase after about 24.16 days.

11.8. Symmetry Relations of Functions

Determine which symmetry relations the following functions $f : D_f \to \mathbb{R}$ have:

1. $f(x) = x^3 - 5x$ **Solution:**
   $f(-x) = (-x)^3 - 5(-x) = -x^3 + 5x = -f(x)$, d.h. $f(x)$ is an odd function.
2. $f(x) = \sqrt{x}$ **Solution:** $D_f = [0; +\infty)$ Since $f(-x) = \sqrt{(-x)}$ is defined only for $x = 0$ on D_f, $f(x)$ is neither even nor odd.
3. $f(x) = \sin(x)$ **Solution:**
   $f(-x) = \sin(-x) = -\sin(x) = -f(x)$, that means that $f(x)$ is an odd function.
4. $f(x) = |x|$ **Solution:** $f(-x) = |-x| = |x| = f(x)$, that means that $f(x)$ is an even function.

5. $f(x) = \sin(x + \pi)$ **Solution:**
$f(-x) = \sin(-x + \pi) = -\sin(x + \pi) = -f(x)$, that means that $f(x)$ is an odd function.

11.9. Translation, Stretching and Compression

1. Perform the following translations and modifications on the unit parabola:

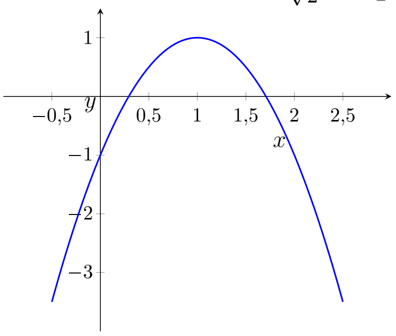
   a) Translation by 1 to the right

   b) Reflection about the x-axis

   c) Stretching by a factor 2 in the $y-$direction

   d) Translation upwards by 1

   Sketch the graph of the function. Calculate the zeros in order to do this. Write the equation for the function in the conventional form. **Solution:** $f(x) = -2\,(x-1)^2 + 1 = -2x^2 + 4x - 1$
   The vertex point is at $S = (1{,}1)$. The zeros are at: $x_{1,2} = 1 \pm \sqrt{1 - \frac{1}{2}} = 1 \pm \dfrac{1}{\sqrt{2}}$
   Using $\sqrt{2} \approx 1\ 4$ and $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ provides the corresponding points for the sketch:



2. Translate the function $f(x) = 3e^{-5x} - 5$ so that the graph intercepts the x-axis at $x = 0$. **Solution:** $f(0) = -2 \Rightarrow f_{neu}(x) = f(x) + 2 = 3e^{-5x} - 3$

3. Stretch the function $f(x) = -e^{3x-3} - 2$ so that it has a value 4 at $x = 1$. **Solution:** $f(1) = -3 \Rightarrow f_{neu}(x) = -\frac{4}{3} \cdot f(x) = \frac{4}{3}e^{3x-3} + \frac{8}{3}$

4. Determine the expression for the resulting function $g(x)$ when

   a) The amplitude of function $f(x) = \sin(x)$ is doubled and the graph of the function is then translated downwards by 3. **Solution:** $g(x) = 2\sin(x) - 3$

   b) The amplitude of function $f(x) = \cos(x)$ is reduced to a third and the period is quadrupled. **Solution:** $g(x) = \frac{1}{3}\cos\left(\frac{x}{4}\right)$

   c) The graph of the function $f(x) = \sin(x)$ is translated upwards by 1 and to the left by 4. **Solution:** $g(x) = \sin(x + 4) + 1$

11.10. Combination and Composition of Functions

1. Determine $(f \circ g)(x)$ and $(g \circ f)(x)$ and the corresponding domains!

   a) $f(x) = 1 + x^2, \quad D_f = \mathbb{R}$ and $g(x) = \sqrt{x}, \quad D_g = [0;\infty)$
   Solution: $(f \circ g)(x) = f(g(x)) = 1 + x, \quad D = [0;\infty)$
   $(g \circ f)(x) = g(f(x)) = \sqrt{1 + x^2}, \quad D = \mathbb{R}$

b) $f(x) = (3 - x)^2, \quad D_f = \mathbb{R} \quad$ and $\quad g(x) = 3x + 1, \quad D_g = \mathbb{R}$
Solution: $(f \circ g)(x) = f(g(x)) = (2 - 3x)^2, \quad D = \mathbb{R}$
$(g \circ f)(x) = g(f(x)) = 3(3 - x)^2 + 1, \quad D = \mathbb{R}$

c) $f(x) = -e^x, \quad D_f = \mathbb{R} \quad$ and $\quad g(x) = x^2 - 3, \quad D_g = \mathbb{R}$
Solution: $(f \circ g)(x) = f(g(x)) = -e^{x^2-3}, \quad D = \mathbb{R}$
$(g \circ f)(x) = g(f(x)) = (-e^x)^2 - 3 = e^{2x} - 3, \quad D = \mathbb{R}$

2. Determine a possible composition of form $h \circ g$ for function f.

a) $f(x) = \sin(x - 3)$ **Solution:** $g(x) = x - 3, \ h(x) = \sin(x)$

b) $f(x) = (2x + 8)^3$ **Solution:** $g(x) = 2x + 8, \ h(x) = x^3$

c) $f(x) = \dfrac{1}{x^2 - 7}$ **Solution:** $g(x) = x^2 - 7, \ h(x) = \dfrac{1}{x}$

d) $f(x) = e^{2x}$ **Solution:** $g(x) = 2x, \ h(x) = e^x$

11.11. For Experienced Students

1. A chess club is hosting a large chess tournament. Each participant plays exactly one match against every other participant. After each match, the players are given cards. The winners are given green cards, the losers are given red cards. In case of a tie, each player receives a yellow card.
After the tournament, the organizer determines that exactly 752 cards of each color were distributed. How many participants took part in the tournament?

Solution: Number of matches: $752 \cdot 3 : 2 = 1128$.
For n players there are $(n-1)+(n-2)+(n-3)+\cdots+3+2+1 = n\cdot(n-1) : 2$ matches.
Therefore $n^2 - n - 2256 = 0$ applies and thus 48 players participated in the tournament.

2. Using the two building blocks $x + 3$ and $x^2 - 4$, construct a rational function

a) of degree 3 with 3 simple zeros. **Solution:** For example, $f(x) = (x+3)(x^2 - 4)$ and every possible stretched version of this function.

b) of degree 4 with 2 double zeros. **Solution:** For example, $f(x) = (x^2 - 4)^2$ and every possible stretched version of this function.

c) of degree 5 with exactly one zero. **Solution:** For example, $f(x) = (x + 3)^5$ and every possible stretched version of this function.

3. Solve.

a) $-2\cos(\pi(x - 1)) + 1 = 2, \ x \in [-2; 2]$ **Solution:** $L = \left\{-\frac{5}{3}; \ -\frac{1}{3}; \ \frac{1}{3}; \ \frac{5}{3}\right\}$

b) $-2\cos(x + \pi) = \sqrt{3}, \ x \in [0; 2\pi]$ **Solution:** $L = \left\{\frac{1}{6}\pi; \ \frac{11}{6}\pi\right\}$

4. Determine the exponential function of form $f(x) = a \cdot q^x$ whose graph goes through the points P(-2/50) and Q(3/0,512).
Solution: P(-2/50) $\Rightarrow a \cdot q^{-2} = 50$, Q(3/0,512) $\Rightarrow a \cdot q^3 = 0\ 512$
$\Rightarrow q^5 = 0\ 01024 \Rightarrow q = \sqrt[5]{0\ 01024} = 0\ 4 \Rightarrow a = 50 \cdot 0\ 4 = 8 \Rightarrow f(x) = 8 \cdot 0\ 4^x$

5. The amount of active ingredient of a pain killer declines approximately exponentially inside a human body. If a patient takes a tablet containing 0.5g of the active ingredient, approximately 0.09g remain after 10 hours.
Solution: Form of the function: $f(t) = 0\ 5 \cdot q^t$, where t represents the number of hours

since the tablet was taken. $f(10) = 0\ 09$ applies.
$\Rightarrow q = \sqrt[10]{0\ 18} \approx 0\ 8424 \Rightarrow f(t) = 0\ 5 \cdot 0\ 8424^t = 0\ 5 \cdot e^{\ln(0\ 8424) \cdot t}$

a) After what period of time has the amount of active ingredient declined by half (by 90%) ?
Solution: The half-life t_H is being asked for, with $f(t_H) = 1/2 \cdot 0\ 5 \Rightarrow t_H \approx 4\ 042$ hours. $f(t) = 0\ 1 \cdot 0\ 5 \Rightarrow t \approx 13\ 43$ hours.

b) A person takes one tablet at 9am and a second tablet at 3pm, each containing 0.5g of the active ingredient. How many grams of the active ingredient are still present in the body at 8pm on the same day?
Solution: $f(t) = 0\ 5 \cdot e^{\ln(0\ 8424) \cdot t}$, $g(t) = 1 \cdot e^{\ln(0\ 8424) \cdot t}$, $f(11) + g(5) \approx 0\ 5$ g.

12. Differential and Integral Calculus

12.1. Differential Calculus

1. Determine the first and second derivatives of the functions.

   a) $f(x) = 5x^5 - 3x^4 - 2x^2 - 3$ **Solution:**

   $$f'(x) = 25x^4 - 12x^3 - 4x, \qquad f''(x) = 100x^3 - 36x^2 - 4$$

   b) $f(x) = 2(3x^3 - 2x)$ **Solution:**

   $$f'(x) = 2(9x^2 - 2), \qquad f''(x) = 36x$$

   c) $f(x) = x^3 \cdot (2x^2 - 4)$ **Solution:**

   $$f'(x) = 3x^2 \cdot (2x^2 - 4) + x^3 \cdot 4x = 10x^4 - 12x^2, \qquad f''(x) = 40x^3 - 24x$$

   d) $f(x) = 2x^3 \cdot 4x^2$ **Solution:**

   $$f'(x) = 40x^4, \qquad f''(x) = 160x^3$$

   e) $f(x) = \dfrac{2x^2 + 3x}{4x^3}$ **Solution:**

   $$\begin{aligned} f'(x) &= \frac{(4x+3) \cdot 4x^3 - (2x^2 + 3x) \cdot 12x^2}{(4x^3)^2} \\ &= \frac{(4x+3) \cdot 4x^3 - (2x+3) \cdot 3 \cdot 4x^3}{(4x^3)^2} \\ &= \frac{-2x - 6}{4x^3} = -\frac{x+3}{2x^3} \\ f''(x) &= \frac{2x + 9}{2x^4} \end{aligned}$$

   f) $f(x) = \dfrac{x^4 - 1}{2x}$ **Solution:**

   $$f'(x) = \frac{3x^4 + 1}{2x^2}, \qquad f''(x) = \frac{3x^4 - 1}{x^3}$$

   g) $f(x) = \dfrac{1}{x+1}$ **Solution:**

   $$f'(x) = -\frac{1}{(x+1)^2}, \qquad f''(x) = \frac{2}{(x+1)^3}$$

   h) $f(x) = (x+1)^7$ **Solution:**

   $$f'(x) = 7(x+1)^6, \qquad f''(x) = 42(x+1)^5$$

i) $f(x) = (x^2-1)^{-3}$ **Solution:**

$$f'(x) = -6x(x^2-1)^{-4} = -\frac{6x}{(x^2-1)^4}$$

$$f''(x) = -6(x^2-1)^{-4} - 6x\cdot(-4)(x^2-1)^{-5}\cdot(2x) = (42x^2+6)\cdot(x^2-1)^{-5} = \frac{42x^2+6}{(x^2-1)^5}$$

Note! The lowest potential of the common factor is factored out. Generally, $z^{-4} = z^{-5}\cdot z$ applies.

2. Derive the following functions.

a) $f(x) = \sqrt{3x-4}$ **Solution:**

$$f'(x) = \frac{3}{2\sqrt{3x-4}}$$

b) $f(x) = (\sin x)^2$ **Solution:**

$$f'(x) = 2\sin x\cos x$$

c) $f(x) = \cos x^2$ **Solution:**

$$f'(x) = -\sin x^2\cdot 2x$$

d) $f(x) = \sqrt[3]{2x^2+3x}$ **Solution:**

$$f'(x) = \frac{4x+3}{3\sqrt[3]{2x^2+3x}^2}$$

e) $f(x) = x^3\cdot\sin x$ **Solution:**

$$f'(x) = 3x^2\cdot\sin x + x^3\cdot\cos x = x^2(3\sin x + x\cos x)$$

f) $f(x) = \dfrac{3x^2}{\cos x}$ **Solution:**

$$f'(x) = \frac{6x\cos x + 3x^2\sin x}{(\cos x)^2}$$

12.2. Applications for Differential Calculus

1. Determine the first derivative of the following functions and find the equation for the tangent at point $x_1 = 2$

a) $f(x) = 2x^4$ **Solution:** $f'(x) = 8x^3$, $\quad t(x) = 64(x-2)+32 = 64x-96$

b) $f(x) = \dfrac{1}{x^2}$ **Solution:** $f'(x) = -\frac{2}{x^3}$, $\quad t(x) = -\frac{1}{4}(x-2)+\frac{1}{4} = -\frac{1}{4}x+\frac{3}{4}$

c) $f(x) = x^{-3}$ **Solution:** $f'(x) = -\frac{3}{x^4}$, $\quad t(x) = -\frac{3}{16}(x-2)+\frac{1}{8} = -\frac{3}{16}x+\frac{1}{2}$

d) $f(x) = \left(\frac{1}{4}x\right)^3$ **Solution:** $f'(x) = \frac{3}{4}\left(\frac{1}{4}x\right)^2$, $\quad t(x) = \frac{3}{16}(x-2)+\frac{1}{8} = \frac{3}{16}x-\frac{1}{4}$

2. Determine the extreme points for the following functions and decide for each respective case whether it is a local maximum or a local minimum.

a) $f(x) = \frac{1}{3}x^3 + \frac{7}{4}x^2 - 2x - 3$ **Solution:** $f'(x) = x^2 + \frac{7}{2}x - 2,\ f''(x) = 2x + \frac{7}{2}$
$\Rightarrow$ relative maximum at $x = -4$ and relative minimum at $x = \frac{1}{2}$

b) $f(x) = e^x + e^{-x}$ **Solution:**
$f'(x) = e^x - e^{-x},\ f''(x) = e^x + e^{-x} \Rightarrow$ relative minimum at $x = 0$

c) $f(x) = 2\sin(\pi x)$ in the interval $[0;2]$ **Solution:** $f'(x) = 2\pi\cos(\pi x),\ f''(x) = -2\pi^2\sin(\pi x)$
$\Rightarrow$ relative maximum at $x = \frac{1}{2}$ and relative minimum at $x = \frac{3}{2}$

12.3. Integral Calculus

1. Calculate $\int f(x)\,dx$ for

a) $f(x) = 2x^4$ **Solution:** Most of the following integrals can be calculated using the rule of integration for the monomial:

First, the exponent is increased by 1 and the multiplied with the reciprocal of the exponent.

It is often first necessary to appropriately transform the equation using the rules for mathematical operations for exponents:

$$\int 2x^4\,dx = \frac{1}{5}\left[2x^5\right] + c = \frac{2}{5}x^5 + c$$

b) $f(x) = \dfrac{1}{x^2}$ **Solution:**

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{1}{-1}\left[x^{-1}\right] + c = -x^{-1} + c = -\frac{1}{x} + c$$

c) $f(x) = \left(\frac{1}{4}x\right)^3$ **Solution:** $\int \left(\frac{1}{4}x\right)^3 dx = \int \frac{1}{64}x^3\,dx = \frac{1}{64}\int x^3\,dx = \frac{1}{64}\left[\frac{1}{4}x^4\right] + c = \frac{1}{256}x^4 + c$

d) $f(x) = 2x^4 + 4x^3 - 3x^2$ **Solution:** $\int \left(2x^4 + 4x^3 - 3x^2\right) dx = \frac{2}{5}x^5 + x^4 - x^3 + c$

e) $f(x) = 5x^5 - 3x^4 - 2x^2 - 3$ **Solution:** $\int \left(5x^5 - 3x^4 - 2x^2 - 3\right) dx = \frac{5}{6}x^6 - \frac{3}{5}x^5 - \frac{2}{3}x^3 - 3x + c$

f) $f(x) = x^3 \cdot (2x^2 - 4)$ **Solution:** $\int x^3 \cdot (2x^2 - 4)\,dx = \int 2x^5 - 4x^3\,dx = \frac{1}{3}x^6 - x^4 + c$

g) $f(x) = 2x^3 \cdot 4x^2$ **Solution:** $\int 2x^3 \cdot 4x^2\,dx = \int 8x^5\,dx = \frac{4}{3}x^6 + c$

h) $f(x) = \dfrac{2x^2 + 3x}{4x^3}$ **Solution:**

$$\int \frac{2x^2 + 3x}{4x^3}\,dx = \int \frac{2x^2}{4x^3} + \frac{3x}{4x^3}\,dx = \int \frac{1}{2}x^{-1} + \frac{3}{4}x^{-2}\,dx = \frac{1}{2}\ln|x| - \frac{3}{4}x^{-1} + c$$

i) $f(x) = \dfrac{x^4 - 1}{2x}$ **Solution:**$\int \frac{x^4-1}{2x}\,dx = \int \frac{1}{2}\left(x^3 - x^{-1}\right) dx = \frac{1}{8}x^4 - \frac{1}{2}\ln|x| + c$

j) $f(x) = \dfrac{1}{x+1}$ **Solution:** $\int \frac{1}{x+1}\,dx = \ln|x+1| + c$

k) $f(x) = (x+1)^7$ **Solution:** $\int (x+1)^7 \, dx = \frac{1}{8}(x+1)^8 + c$

l) $f(x) = \sqrt{x-4}$ **Solution:**

$$\int \sqrt{x-4}\, dx = \int (x-4)^{\frac{1}{2}} \, dx = \frac{2}{3}(x-4)^{\frac{3}{2}} + c = \frac{2}{3}\sqrt{x-4}^3 + c$$

m) $f(x) = \sin x$ **Solution:** $\int \sin x \, dx = -\cos x + c$

n) $f(x) = \dfrac{3}{\sqrt[5]{x}}$ **Solution:**

$$\int \frac{3}{\sqrt[5]{x}} \, dx = \int 3x^{-\frac{1}{5}} \, dx = \frac{15}{4} x^{\frac{4}{5}} + c = \frac{15}{4}\sqrt[5]{x^4} + c$$

o) $f(x) = \sqrt{x}\,(\sqrt{x}-1)^2$ **Solution:**

$$\begin{aligned}\int \sqrt{x}\left(\sqrt{x}-1\right)^2 \, dx &= \int \sqrt{x}\left(x - 2\sqrt{x} + 1\right) \, dx = \int x^{\frac{3}{2}} - 2x + x^{\frac{1}{2}} \, dx \\ &= \frac{2}{5}x^{\frac{5}{2}} - x^2 + \frac{2}{3}x^{\frac{3}{2}} + c = \frac{2}{5}\sqrt{x^5} - x^2 + \frac{2}{3}\sqrt{x^3} + c\end{aligned}$$

2. Calculate **Solution:**

a) $\int_0^1 (x - x^2) \, dx = \left[\frac{1}{2}x^2 - \dfrac{1}{3}x^3\right]_0^1 = \dfrac{1}{2} - \dfrac{1}{3} = \dfrac{1}{6}$

b)

$$\begin{aligned}\int\limits_1^4 2(3x^3 - 2x)dx &= \left[\frac{3}{2}x^4 - 2x^2\right]_1^4 = \frac{3}{2}4^4 - 2\cdot 4^2 - \left(\frac{3}{2}1^4 - 2\cdot 1^2\right) \\ &= \frac{3}{2}\cdot 256 - 32 - \frac{3}{2}1^4 + 2 = 354 - \frac{3}{2} = \frac{705}{2}\end{aligned}$$

c)

$$\begin{aligned}\int\limits_{\frac{1}{5}}^{5} x^{-4} dx &= \left[-\frac{1}{3}x^{-3}\right]_{\frac{1}{5}}^{5} = -\frac{1}{3}5^{-3} + \frac{1}{3}5^3 = \frac{1}{3}\left(125 - \frac{1}{125}\right) \\ &= \frac{1}{3}\left(\frac{(12500 + 2500 + 625) - 1}{125}\right) = \frac{1}{3}\cdot\frac{15624}{125} = \frac{5208}{125}\end{aligned}$$

3. Calculate the area underneath the curve of $f : \mathbb{R} \to \mathbb{R}$ for $x \geqslant 0$, $f(x) \geqslant 0$ where

a) $f(x) = 4 - x^2$
Solution: The lower integration limit is 0 because $x \geqslant 0$ and the upper limit is 2 because: $4 - x^2 \geqslant 0$ for $-2 \leqslant x \leqslant 2$. Thus it follows that:

$$\int_0^2 \left(4 - x^2\right) \, dx = \left[4x - \frac{1}{3}x^3\right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$$

b) $f(x) = e - e^x$

Solution: The lower integration limit is 0 because $x \geqslant 0$ and the upper limit is 1 because: $e - e^x \geqslant 0$ for $x \leqslant 1$ Thus it follows that: $\int_0^1 (e - e^x)\; dx = [ex - e^x]_0^1 = e - e + 1 = 1$

12.4. For Experienced Students

1. Determine a polynomial function of the lowest degree possible for which the following applies:

   a) The function has a maximum at H=(0;1).

   b) The graph of the function intercepts the x-axis at $x = 2$.

   c) There is a point of inflection at $x = 1$.

   Solution: The conditions result in 4 equations. The function must therefore be at least third order. That means $f(x) = ax^3 + bx^2 + cx + d$.

   a) $f(0) = 1$ and $f'(0) = 0$ b) $f(2) = 0$ c) $f''(1) = 0$

   This results in a system of equations with 4 unknowns: a, b, c and d.
   $\Rightarrow a = \frac{1}{4},\; b = -\frac{3}{4},\; c = 0,\; d = 1$ and therefore the function $f(x) = \frac{1}{4}x^3 - \frac{3}{4}x^2 + 1$.

2. Which function $f(x) = a \cdot \sin(\pi \cdot x) + b$ has slope 3π and value 4 at $x = 2$?
   Solution: $f'(x) = a \cdot \pi \cdot \cos(\pi \cdot x)$ applies. The conditions result in the equations $a \cdot \pi \cdot \cos(\pi \cdot 2) = 3 \cdot \pi,\; a \cdot \sin(\pi \cdot 2) + b = 4$ and therefore $f(x) = 3 \cdot \sin(\pi \cdot x) + 4$.

3. Determine a, b and k for $f(x) = a \cdot e^{kx} + b$ under the following conditions:

   a) The asymptote of the graph of the function is described by the equation $y = 3$.

   b) The graph of the function intercepts the y-axis at -2 .

   c) The tangent at point $x = 0$ has slope $5 \cdot \ln(2)$.

   Solution:

   a) $f(x) \to 3$ for $x \to +\infty$. $\Rightarrow b = 3$

   b) $f(0) = -2 \;\Rightarrow a + b = -2$

   c) $f'(0) = 5 \cdot \ln(2) \Rightarrow a \cdot k + b = 5 \cdot \ln(2)$

   $\Rightarrow a = -5,\; b = 3,\; k = -\ln(2)$ and therefore the function $f(x) = -5e^{-ln(2)x} + 3$.

13. Vector Algebra

13.1. Geometry Equations

Calculate the volume as well as the surface area

1. of the cuboid with edge lengths: L= 100 m; B= 100 cm; H= 50 mm.
   Solution: $V = 5\text{m}^3$, $A = L \cdot B \cdot 2 + 2 \cdot L \cdot H + 2 \cdot B \cdot H = 210\ 1\text{m}^2$

2. the cylinder with diameter D= 5 km and height H= 500 mm.
   Solution: $V = 9\ 8 \cdot 10^6\text{m}^3$, $A = 39277762\text{m}^2$

13.2. Mathematical Operations with Vectors

1. A straight prism ABCDEF, has A, B, and C as the corners of its base. The height of the prism is 5. Determine the coordinates of points D, E, and F if

   a) A=(2;0;3), B=(1;0;7), C=(-7;0;3) **Solution:** D=(2;5;3), E=(1;5;7), F=(-7;5;3)

   b) A=(2;0;3), B=(6;2;3), C=(3;3;3) **Solution:** D=(2;0;8), E=(6;2;8), F=(3;3;8)

   c) Which special position do points A, B, and C have in the coordinate system? **Solution:** The base area of the first prism is in the x_1x_3-plane and the base area of the second prism is in a plane parallel to the x_1x_2-plane which has shifted upwards by 3 length units.

2. Calculate with the following vectors and scalars:

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \vec{v} = \begin{pmatrix} \frac{3}{2} \\ -3 \\ 0 \end{pmatrix}, \vec{w} = \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix}, k = 5, l = -2$$

   Solution:

   a)

$$\vec{u} + \vec{v} + \vec{w} = \begin{pmatrix} \frac{13}{2} \\ -1 \\ 9 \end{pmatrix}, \vec{u} - \vec{v} - \vec{w} = \begin{pmatrix} -\frac{9}{2} \\ 5 \\ -3 \end{pmatrix}, \vec{u} - \vec{v} + \vec{w} = \begin{pmatrix} \frac{7}{2} \\ 5 \\ 9 \end{pmatrix}$$

   b)

$$l\vec{w} = \begin{pmatrix} -8 \\ 0 \\ -12 \end{pmatrix}, k\vec{u} + l\vec{v} = \begin{pmatrix} 2 \\ 16 \\ 15 \end{pmatrix}, k\vec{v} - l\vec{v} = \begin{pmatrix} \frac{21}{2} \\ -21 \\ 0 \end{pmatrix}$$

   c) a, b, so that $a\vec{u} + b\vec{v} = \vec{w}$ holds. What does the result tell you? **Solution:**

$$\begin{aligned} 1a + \frac{3}{2}b &= 4 \\ 2a - 3b &= 0 \\ 3a + 0 &= 6 \Rightarrow a = 2,\ b = \frac{4}{3} \end{aligned}$$

   Note: The solution must fulfill all three equations. This requirement needs to be

verified!
The three vectors are linearly dependent. Each pairing of two vectors spans a plane to which the third vector is collinear. They are not pairwise linearly dependent. (Take the zero elements into account.)

d) The length of all three vectors.
Solution:

$$|\vec{u}| = \sqrt{14},\ |\vec{v}| = \frac{3}{2}\sqrt{5},\ |\vec{w}| = 2\sqrt{13}$$

3. Use vectors to determine the midpoint of segment AB.
Solution: $M = A + \frac{1}{2}\overrightarrow{AB}$

   a) A=(3;2;5), B=(5;2;3) **Solution:** M=(4;2;4)

   b) A=(2;1;-2), B=(-5;1;9) **Solution:** M=(-1.5;1;3.5)

   c) A=(0;0;2), B=(-2;0;0) **Solution:** M=(-1;0;1)

4. For triangle ABC, pointsM_A, M_B and M_C are the midpoints of the triangle edges which lie directly opposite the respective corners. Determine the coordinates of points M_A, M_B and M_C as well as the sum of the vectors $\overrightarrow{AM_A}$, $\overrightarrow{BM_B}$ and $\overrightarrow{CM_C}$.
Solution:

$$\overrightarrow{OM_A} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC}, \overrightarrow{OM_B} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}, \overrightarrow{OM_C} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BA}$$

   a) A=(0;0), B=(3;1), C=(1;3) **Solution:**
   $M_A = (2;2)$, $M_B = (0{,}5;1\ 5)$, $M_C = (1\ 5;0{,}5)$, $\overrightarrow{AM_A} + \overrightarrow{BM_B} + \overrightarrow{CM_C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

   b) A=(0;0;0), B=(3;1;2), C=(1;3;4) **Solution:**
   $M_A = (2;2;3)$, $M_B = (0\ 5;1\ 5;2)$, $M_C = (1\ 5;0\ 5;1)$, $\overrightarrow{AM_A} + \overrightarrow{BM_B} + \overrightarrow{CM_C} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

5. Points P=(1;3;5) and Q=(2;-1;7) are given.

   a) Determine at least 2 different equations for the line through P and Q.
   Solution: Possible equations could be $g_1 : \vec{x} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + r \cdot \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix},\ r \in \mathbb{R},\ \ g_2 : \vec{x} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + s \cdot \begin{pmatrix} -2 \\ 8 \\ -4 \end{pmatrix},\ s \in \mathbb{R}$ or $g_3 : \vec{x} = \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix},\ t \in \mathbb{R}$

   b) Determine whether point R=(2;-5;9) lies on the line. **Solution:** R does not line on the line. If the point is substituted into one of the equations for a line, for example into g_1, then solving for the parameter r in the first line delivers $r = 1$ and in lines 2 and 3 $r = 2$.

6. Points A=(7;5;4), B=(-5;8;7) and C=(-1;1;3) are given.

   a) Show that triangle ABC is an isosceles right triangle.

Solution:

$$\overrightarrow{AB} = \begin{pmatrix} -12 \\ 3 \\ 3 \end{pmatrix}, \ \overrightarrow{AC} = \begin{pmatrix} -8 \\ -4 \\ -1 \end{pmatrix}, \ \overrightarrow{BC} = \begin{pmatrix} 4 \\ -7 \\ -4 \end{pmatrix},$$

$|\overrightarrow{AB}| = \sqrt{162}$, $|\overrightarrow{AC}| = \sqrt{81} = 9$, $|\overrightarrow{BC}| = \sqrt{81} = 9$ $\Rightarrow$ the lengths of vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are equal.
$\overrightarrow{AC} \cdot \overrightarrow{BC} = -8 \cdot 4 + 4 \cdot 7 + 1 \cdot 4 = 0$ $\Rightarrow$ right angle at C.

b) Calculate the area of the triangle. **Solution:** $\frac{1}{2}|\overrightarrow{AC}| \cdot |\overrightarrow{BC}| = \frac{1}{2} \cdot 9 \cdot 9 = 40\ 5$ area units.

7. Determine the relative position of line g to line h.

$$g : \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + r \cdot \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}, \ r \in \mathbb{R}, \ h : \vec{x} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix}, \ s \in \mathbb{R}$$

Solution: The lines are parallel to one another but not identical.

8. Determine the intersection of lines g and h.

$$g : \vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + r \cdot \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}, \ r \in \mathbb{R}, \ h : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + s \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \ s \in \mathbb{R}$$ **Solution:** S=(3;1;5)

9. Given the vectors $\vec{a} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$.

a) Calculate the cross products $\vec{a} \times \vec{b}$, $\vec{a} \times \vec{c}$ and $\vec{b} \times \vec{c}$.

Solution: $\vec{a} \times \vec{b} = \begin{pmatrix} 2 \cdot 2 - 1 \cdot 7 \\ 1 \cdot 1 + 1 \cdot 2 \\ -1 \cdot 7 - 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -9 \end{pmatrix}$

$$\vec{a} \times \vec{c} = \begin{pmatrix} 2 \cdot 5 - 1 \cdot 2 \\ 1 \cdot 2 + 1 \cdot 5 \\ -1 \cdot 2 - 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \\ -6 \end{pmatrix}, \ \vec{b} \times \vec{c} = \begin{pmatrix} 7 \cdot 5 - 2 \cdot 2 \\ 2 \cdot 2 - 1 \cdot 5 \\ 1 \cdot 2 - 7 \cdot 2 \end{pmatrix} = \begin{pmatrix} 31 \\ -1 \\ -12 \end{pmatrix}$$

b) Calculate the respective areas of the parallelograms whose adjacent sides are $\vec{a}$ and $\vec{b}$, $\vec{a}$ and $\vec{c}$, as well as $\vec{b}$ and $\vec{c}$, respectively. **Solution:**
This calculation merely requires determining the magnitude of the cross product.

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{9 + 9 + 81} = \sqrt{99} \approx 10\ , \ |\vec{a} \times \vec{c}| = \sqrt{64 + 49 + 36} = \sqrt{149} \approx 12\ 2$$
$$\left|\vec{b} \times \vec{c}\right| = \sqrt{961 + 1 + 144} \approx 33\ 3$$

c) Determine the angle enclosed by $\vec{a}$ and $\vec{b}$. Use the scalar product to do so. Subsequently verify your result by using the cross product.
Solution: Applying the scalar product:

$$\cos\alpha = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{15}{\sqrt{6} \cdot \sqrt{54}} = \frac{5}{6} \Rightarrow \alpha = 33\ 6^\circ$$

Verifying by using the cross product:

$$\sin\alpha = \frac{\left|\vec{a} \times \vec{b}\right|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{99}}{\sqrt{6} \cdot \sqrt{54}} \Rightarrow \alpha = 33\ 6^\circ$$

Analogously, calculated the angle between $\vec{a}$ and $\vec{c}$, as well as between $\vec{b}$ and $\vec{c}$.
Solution: Angle between $\vec{a}$ and $\vec{c}$

$$\cos\beta = \frac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{c}|} = \frac{7}{\sqrt{6}\cdot\sqrt{33}} \Rightarrow \beta = 60,\ 2$$

$$\sin\beta = \frac{|\vec{a}\times\vec{c}|}{|\vec{a}||\vec{c}|} = \frac{\sqrt{149}}{\sqrt{6}\cdot\sqrt{33}} \Rightarrow \beta = 60\ 2^\circ$$

Angle between $\vec{b}$ and $\vec{c}$

$$\cos\gamma = \frac{\vec{b}\cdot\vec{c}}{|\vec{b}||\vec{c}|} = \frac{26}{\sqrt{54}\cdot\sqrt{33}} \Rightarrow \gamma = 52^\circ$$

$$\sin\gamma = \frac{\left|\vec{b}\times\vec{c}\right|}{|\vec{b}||\vec{c}|} = \frac{\sqrt{1106}}{\sqrt{54}\cdot\sqrt{33}} \Rightarrow \gamma = 52^\circ$$

13.3. For Experienced Students

The straight line flight paths of two airplanes F_1 and F_2 can be specified using a coordinate system. Flight path F_1 is given by the points P=(2;3;1) and Q=(0;0;1,05) and flight path F_2 is given by R=(-2;3;0,05) and T=(2;-3;0,07). The coordinates specify the distances from the coordinate origin in kilometers. There is no wind. F_1 flies with a velocity of 350 km/h and F_2 with a velocity of 250 km/h relative to the air. The location of F_1 is at point P and at the same time the location of F_2 is at point R. Let us consider the situation 20 minutes later.

Solution: The flight paths can be described by the equations for a line $f_1 : \vec{x} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + r \cdot \begin{pmatrix} -2 \\ -3 \\ 0{,}05 \end{pmatrix}$, $r \in \mathbb{R}$ and $f_2 : \vec{x} = \begin{pmatrix} -2 \\ 3 \\ 0{,}05 \end{pmatrix} + s \cdot \begin{pmatrix} 4 \\ -6 \\ 0{,}02 \end{pmatrix}$, $s \in \mathbb{R}$.

1. What is the location of the two airplanes? What is their altitude?
   Solution: In 20 minutes, F_1 flies $\frac{350}{3}$km and F_2 flies $\frac{250}{3}$km.
   The magnitudes of the direction vectors are $\left|\begin{pmatrix} -2 \\ -3 \\ 0\ 05 \end{pmatrix}\right| = \sqrt{4+9+0\ 0025} \approx 3\ 6$ for the flight path of F_1 and $\left|\begin{pmatrix} 4 \\ -6 \\ 0\ 02 \end{pmatrix}\right| = \sqrt{16+36+0\ 0004} \approx 7\ 2$ for the flight path of F_2.
   Therefore the parameter values are $r = \frac{350}{3} : 3\ 6 = 32\ 4$ and $s = \frac{250}{3} : 7\ 2 = 11\ 6$. The coordinates of F_1 are therefore (-62.8;-94.2;2.62) and those of F_2 are (44.4;-66.6;0.282). F_1 is at an altitude of approximately 2620 m and F_2 is at a altitude of approximately 282 m.

2. How far are the two airplanes apart?
   Solution: They are approximately 110.7 km apart.

14. Stochastic Processes

14.1. Relative and Absolute Frequencies

1. After an inspection of 2325 cattle, the relative frequency of animals infected with brucellosis was determined to be 0.04.

   a) How many of the inspected animals were infected with the bacteria? **Solution:** 93 animals

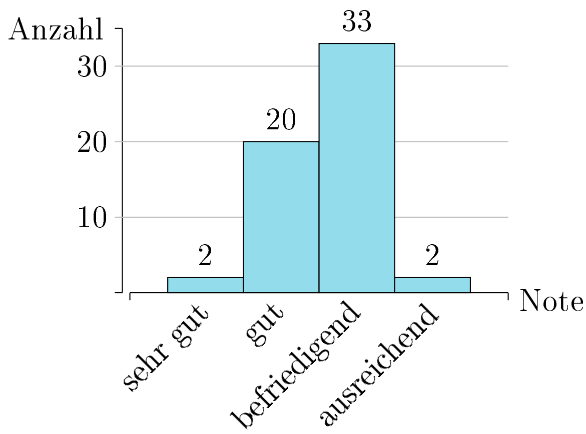
   b) How exact is the value you have calculated? **Solution:** The calculated number of animals is exactly correct.

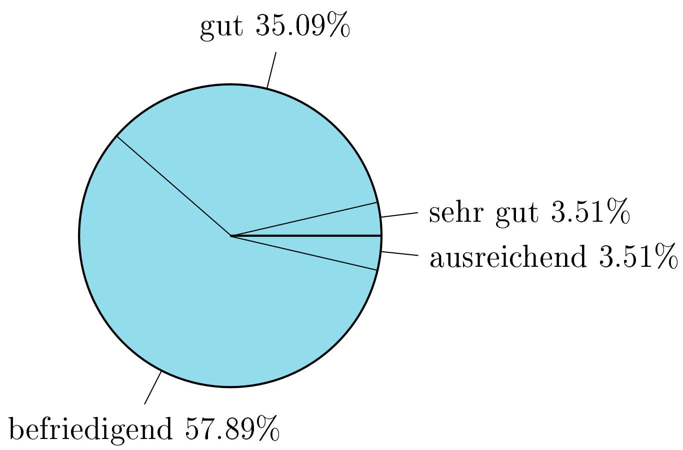
2. In a random sample of 57 high school graduates, the attribute "final grade point average" was compiled.

   a) Determine the relative frequencies associated with the attribute classes very good $(1\,0 \leqslant x \leqslant 1\,5)$, good $(1\,6 \leqslant x \leqslant 2\,5)$, satisfactory $(2\,6 \leqslant x \leqslant 3\,5)$ and sufficient $(3\,6 \leqslant x \leqslant 4\,0)$. Create a tally chart in order to do this. **Solution:**

Class	very good	good	satisfactory	sufficient
Tally list	||	卌 卌 卌 卌	卌 卌 卌 卌 卌 卌 |||	||
absolute frequency	2	20	33	2
relative frequency	3.51 %	35.09 %	57.89 %	3.51 %

   b) Display the distribution of the absolute frequencies in a bar chart and the distribution of the relative frequencies in a pie chart.. **Solution:**





3. One third of the employees of a certain company earns € 1600 per month, one fifth earns € 2000 per month, one sixth earns € 2300 per month and the rest earns € 3000 per month.

   a) Determine the average monthly income.
   Solution: $\bar{x} = \frac{1}{3} \cdot 1600 + \frac{1}{5} \cdot 2000 + \frac{1}{6} \cdot 2300 + (1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{6}) \cdot 3000 = 2216{,}67$ Euros

   b) What is the standard deviation?
   Solution: $\sigma = 566\ 90$ Euros

4. In order to obtain an indication of the average annual mileage of passenger vehicles, 50 auto owners were randomly selected and asked about the total distance (in 1000s of km) they drove in the past year. following table contains the initial results.

15	12	16	25	5	30	8	10	15	20	7	10	20	30	15	25	21
15	18	30	45	5	2	16	24	25	10	14	20	15	12	8	14	5
25	10	15	28	12	10	20	32	42	13	18	25	12	15	20	2	

Determine the arithmetic average, the variance, and the standard deviation. **Solution:**

- arithmetic average: $\bar{x} = 17\ 22$
- variance: $\sigma^2 = 84\ 89$
- standard deviation: $\sigma = 9\ 21$

14.2. Arithmetic with Probabilities

1. According to a statistic presented by the German rail company, approximately 95 percent of long-distance trains run "on time", that is with a maximum of 5 minutes delay. Tim travels five times by long-distance train.

   a) He calculates the probability that at least one train is not on time using the formula $1 - 0\ 95^5$. Under what circumstances can he apply that formula?
   Solution: Tim assumes that the punctuality of different trains is independent from one another. He calculates the complement of "all trains are punctual", ergo "at least one train is not punctual".

   b) Why is the assumption which Tim makes not necessarily correct? **Solution:**
   The punctuality of trains is not independent. A connecting train waits, defective trains can in some circumstances not be passed, etc.

2. The letters M, U, and T are to be used to spell three-letter words in which each letter appears only once. The words do not have to make sense. We consider the event E: "There is a consonant at the beginning" and F: "The last letter is a T".

   a) Describe the events E and F as sets and determine their probabilities. **Solution:**
   $E = \{MUT, MTU, TMU, TUM\}, \quad P(E) = \frac{2}{3}$,
   $F = \{MUT, UMT\}, \quad P(F) = \frac{1}{3}$

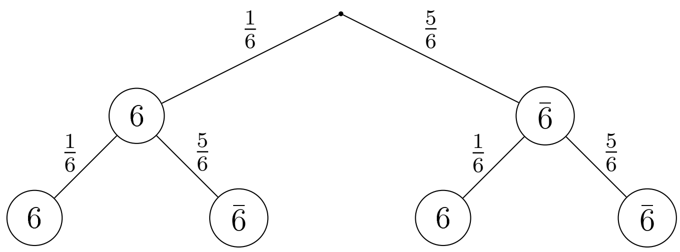
   b) Describe the events $E \cap F$ and $E \cup F$ with words and determine their probabilities.
   Solution:
   $E \cap F$: "There is a consonant at the beginning and the last letter is a T"
   $E \cap F = \{MUT\}$ and $P(E \cap F) = \frac{1}{6}$.
   $E \cup F$: "There is a consonant at the beginning or the last letter is a T".
   $E \cap F = \{MUT, UMT, TUM, TMU, MTU\}$ and $P(E \cup F) = \frac{5}{6}$

c) Analyze whether E and F are independent events. **Solution:**
It is $P(E) \cdot P(F) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$ and $P(E \cap F) = \frac{1}{6}$, that means that whether or not there is a T at the end influences the probability of whether there is a consonant at the beginning and vice versa.

3. A coin is tossed until one side appears for the second time. Determine the probability distribution for the random variable X: "Number of tosses" as well as the expected value and the standard deviation of X. **Solution:**

   - If $X = 2$ applies, the solution set is $\{WW, ZZ\}$ and $P(X = 2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
   - For $X = 3$ we have:$\{WZW, ZWW, WZZ, ZWZ\}$ and $P(X = 3) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$
   - Expected value: $\mu = 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = \frac{5}{2}$
   - Standard deviation: $\sigma = \sqrt{(2 - \frac{5}{2})^2 \cdot \frac{1}{2} + (3 - \frac{5}{2})^2 \cdot \frac{1}{2}} = \frac{1}{2\sqrt{2}}$

4. A student states: "If I roll a die twice, the probability is $\frac{1}{3}$ that the outcomes will include a six because for one roll of a die the probability is $\frac{1}{6}$." Is the student correct? **Solution:** He is not correct, the rolls are independent of one another. Otherwise there would definitely be a six among six rolls. The solution with the assistance of a tree diagram:



Applying the arithmetic rules leads to the probability

$$P(\text{"At least one sic"}) = \frac{11}{36}.$$

The solution can also be calculated by applying the rule of sum.

$$P(\text{"At least one six"}) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$$

5. How high is the probability of rolling at least one six if six dice are being rolled? Where do you apply the independence of events in this calculation? **Solution:** The rolling of a six is independent for individual rolls, that means not rolling a six has the probability $\frac{5}{6}$.

$$P(\text{"At least one six"}) = 1 - \left(\frac{5}{6}\right)^6 \approx 66\ 5\%$$

6. A country performs a statistical survey. The following results are determined for the population: The percentage of employed inhabitants is 60%. The percentage of politically interested inhabitants is 56%. The percentage of politically interested inhabitants who are not employed is 14%.

   a) Calculate the missing percentages in the table. **Solution:**

Grouping	employed	not employed	sum
politically interested	0.42	0.14	0.56
not politically interested	0.18	0.26	0.44
sum	0.60	0.4	1

b) One person is selected at random from all the politically interested inhabitants. What is the probability that that person is employed? **Solution:**

$$P(B|I) = \frac{P(B \cap I)}{P(I)} = \frac{0\ 42}{0\ 56} = \frac{3}{4} = 0\ 75$$

c) One person is selected at random from all the employed inhabitants. What is the probability that that person is politically interested? **Solution:**

$$P(I|B) = \frac{P(B \cap I)}{P(B)} = \frac{0\ 42}{0\ 60} = \frac{7}{10} = 0\ 7$$

14.3. For Experienced Students

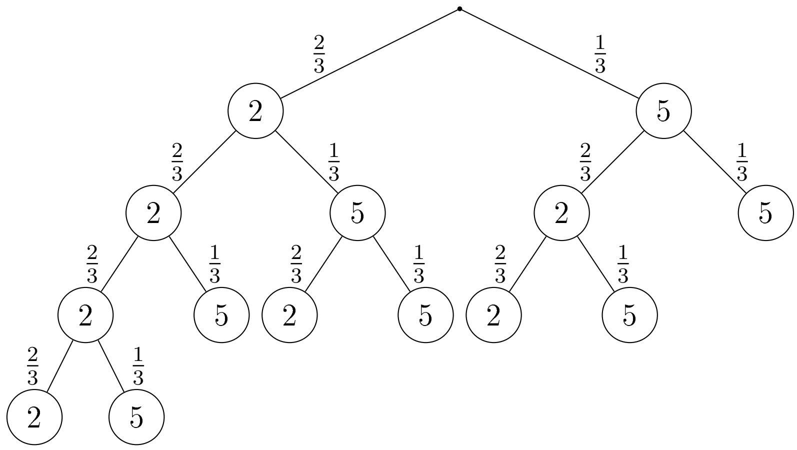
1. There are two possible types of errors which can occur during the production of USB sticks: A defective memory chip (Ch) or a defective connector (Co). The probability of a connector being defective is 5%. The probability of a memory chip being defective is 20%.

    a) Create a fourfold contingency table representing the situation. **Solution:** The provided information is not sufficient to provide a unique solution for the resulting system of equations. Possible fourfold contingency table:

Probabilities			
	no. of defective	no. of okay	sum
no. of defective	0.01	0.19	0.2
no. of okay	0.04	0.76	0.8
sum	0.05	0.95	1

    b) What is the probability of producing a functional USB stick which has neither a defective memory chip nor a defective connector?
    Solution: The fourfold contingency table provides the result $P(E) = 0{,}76$

2. A die has two pips on four of its sides and five pips each on the remaining two sides. This die is rolled repeatedly until the sum of the rolls is at least eight. The random variable X describes the number of rolls required for this to occur. Use a tree diagram to calculate the values of the probability distribution of X and the average number of required rolls.
Solution:



X	2	3	4
$P(X = x_i)$	$\left(\frac{1}{3}\right)^2$	$2 \cdot \left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^2$	$\left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^3 \cdot \frac{1}{3}$
	$= \frac{1}{9}$	$= \frac{16}{27}$	$= \frac{8}{27}$

The resulting average required number of rolls is equal to
$\mu = 2 \cdot \frac{1}{9} + 3 \cdot \frac{16}{27} + 4 \cdot \frac{8}{27} = \frac{86}{27} \approx 3\ 19$